Instructor's Solution Manual

INTRODUCTION TO REAL ANALYSIS

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Contents

CHAPTER 1
THE REAL NUMBERS

1.1 THE REAL NUMBER SYSTEM

1.1.1. Note that $|a-b| = \max(a,b) - \min(a,b)$.

(a) $a+b+|a-b| = a+b+\max(a,b)-\min(a,b) = 2\max(a,b)$.

(b) $a+b-|a-b| = a+b-\max(a,b)+\min(a,b) = 2\min(a,b)$.

(c) Let $\alpha = a+b+2c+|a-b|+\big|a+b-2c+|a-b|\big|$. From **(a)**, $\alpha = 2\left[\max(a,b)+c+|\max(a,b)-c|\right] =_{\rm df} \beta$. From **(a)** with a and b replaced by $\max(a,b)$ and c, $\beta = 4\max\left(\max(a,b),c\right) = 4\max(a,b,c)$.

(d) Let $\alpha = a+b+2c-|a-b|-\big|a+b-2c-|a-b|\big|$. From **(b)**, $\alpha = 2\left[\min(a,b)+c-|\min(a,b)-c|\right] =_{\rm df} \beta$. From **(a)** with a and b replaced by $\min(a,b)$ and c, $\beta = 4\min\left(\min(a,b),c\right) = 4\min(a,b,c)$.

1.1.2. First verify axioms A-E:

<u>Axiom A.</u> See Eqns. (1.1.1) and (1.1.2).

<u>Axiom B.</u> If $a=0$ then $(a+b)+c = b+c$ and $a+(b+c) = b+c$, so $(a+b)+c = a+(b+c)$. Similar arguments apply if $b=0$ or $c=0$. The remaining case is $a=b=c=1$. Since $(1+1)+1 = 0+1 = 1$ and $1+(1+1) = 1+0 = 1$, addition is associative. Since

$$(ab)c = a(bc) = \begin{cases} 0, & \text{unless } a=b=c=1, \\ 1, & \text{if } a=b=c=1, \end{cases}$$

multiplication is associative.

<u>Axiom C.</u> Since

$$a(b+c) = ab+ac = \begin{cases} 0, & \text{if } a=0, \\ b+c, & \text{if } a=1, \end{cases}$$

the distributive law holds.

<u>Axiom D.</u> Eqns. (1.1.1) and (1.1.2) imply that 0 and 1 have the required properties.

Axiom E. Eqn. (1.1.1) implies that $-0 = 0$ and $-1 = 1$; Eqn. (1.1.2) implies that $1/1 = 1$. To see that the field cannot be ordered suppose that $0 < 1$. Adding 1 to both sides yields $0 + 1 < 1 + 1$, or $1 < 0$, a contradiction. On the other hand, if $1 < 0$, then $1 + 1 < 0 + 1$, so $0 < 1$, also a contradiction.

1.1.3. If $\sqrt{2}$ is rational we can write $\sqrt{2} = m/n$, where either m is odd or n is odd. Then $m^2 = 2n^2$, so m is even; thus, $m = 2m_1$ where m_1 is an integer. Therefore, $4m_1^2 = 2n^2$, so $n^2 = 2m_1^2$ and n is also even, a contradiction.

1.1.4. If $\sqrt{p}$ is rational we can write $\sqrt{p} = m/n$ where either m or n is not divisible by p. Then $m^2 = pn^2$, so m is divisible by p; thus, $m = pm_1$ where m_1 is an integer. Therefore, $p^2m_1^2 = pn^2$, so $n^2 = pm_1^2$ and n is also divisible by p, a contradiction.

1.1.6. If S is bounded below then T is bounded above, so T has a unique supremum, by Theorem 1.1.3. Denote $\sup T = -\alpha$. Then (i) if $x \in S$ then $-x \le -\alpha$, so $x \ge \alpha$; (ii) if $\epsilon > 0$ there is an $x_0 \in T$ such that $-x_0 < -\alpha - \epsilon$, so $x_0 > \alpha + \epsilon$. Therefore, there is an α with properties (i) and (ii). If (i) and (ii) hold with α replaced by α_1 then $-\alpha_1$ is a supremum of T, so $\alpha_1 = \alpha$ by the uniqueness assertion of Theorem 1.1.3.

1.1.7. **(a)** If $x \in S$, then $\inf S \le x \le \sup S$, and the transitivity of $\le$ implies (A), with equality if and only if S contains exactly one point.

(b) There are three cases: (i) If S is bounded below and unbounded above, then $\inf S = \alpha$ (finite) and $\sup S = \infty$ from (13); (ii) If S is unbounded below and bounded above, then $\inf S = -\infty$ from (14) and $\sup S = \beta$ (finite); (iii) If S is unbounded below and above then $\sup S = \infty$ from (13) and $\inf S = -\infty$ from (14). In all three cases (12) implies (A).

1.1.8. Let $a = \inf T$ and $b = \sup S$. We first show that $a = b$. If $a < b$ then $a < (a+b)/2 < b$. Since $b = \sup S$, there is an $s_0 \in S$ such that $s_0 > (a+b)/2$. Since $a = \inf T$, there is a $t_0 \in T$ such that $t_0 < (a+b)/2$. Therefore, $t_0 < s_0$, a contradiction. Hence $a \ge b$. If $a > b$ there is an x such that $b < x < a$. From the definitions of a and b, $x \notin T$ and $x \notin S$, a contradiction. Hence $a = b$. Let $\beta = a (= b)$. Since a and b are uniquely, defined so is β. If $x > \beta$ then $x \notin S$ (since $\beta = \sup S$), so $x \in T$. If $x < \beta$ then $x \notin T$ (since $\beta = \inf T$), so $x \in S$.

1.1.9. Every real number is in either S or T. T is nonempty because U is bounded. S is nonempty because if $u \in U$ and $x < u$ then $x \in S$. If $s \in S$ there is a $u \in U$ such that $u > s$, since s is not an upper bound of U. If $t \in T$ then $t \ge u$, since t is an upper bound of U. Since $u > s$ and $t \ge u$, $t > s$. Therefore, S and T satisfy the conditions imposed in Exercise 1.1.8, so there is a number β such that every number greater than β is an upper bound of U and no number less that β is an upper bound of U. However, β is also an upper bound of U (if not, there would be a $u_0 \in U$ such that $u_0 > \beta$, which is impossible, since if $u_0 > \beta$ then every number in (β, u_0) is an upper bound of U). Therefore, $\beta = \sup U$.

1.1.10. **(a)** Let $\alpha = \sup S$ and $\beta = \sup T$. If $x = s + t$ then $x \le \alpha + \beta$. If $\epsilon > 0$ choose s_0 in S and t_0 in T such that $s_0 > \alpha - \epsilon/2$ and $t_0 > \beta - \epsilon/2$. Then $x_0 = s_0 + t_0 > \alpha + \beta - \epsilon$ and $\alpha + \beta = \sup(S + T)$ by Theorem 1.1.3. This proves (A). The proof of (B) by mean of Theorem 1.1.8 is similar.

(b) For (A), suppose that S is not bounded above. Then $\sup S = \sup(S + T) = \infty$. Since $\sup T > -\infty$ if T is nonempty, (A) holds. The proof of (B) is similar.

1.1.11. Apply Exercise 1.1.10 with S and T replaced by S and $-T$.

1.1.12. If $a = 0$ then $T = \{b\}$, so $\inf T = \sup T = b$.

Now suppose that $a \neq 0$ and let $\alpha = \inf S$ and $\beta = \sup S$. From the definitions of α and β,

$$\alpha \le s \quad \text{and} \quad s \le \beta \quad \text{for all} \quad s \in S, \tag{A}$$

and if $\epsilon > 0$ is given there are elements s_1 and s_2 of S such that

$$s_1 < \alpha + \epsilon/|a| \quad \text{and} \quad s_2 > \beta - \epsilon/|a|. \tag{B}$$

CASE 2. If $a > 0$, multiplying the inequalities in (A) by a shows that

$$a\alpha \le as \quad \text{and} \quad as \le a\beta \quad \text{for all} \quad s \in S.$$

Therefore,

$$a\alpha + b \le as + b \quad \text{for all} \quad s \in S, \tag{C}$$

$$as + b \le a\beta + b \quad \text{for all} \quad s \in S. \tag{D}$$

Multiplying the inequalities in (B) by a shows that

$$as_1 < a\alpha + \epsilon \quad \text{and} \quad as_2 > a\beta - \epsilon,$$

since $|a| = a$. Therefore,

$$as_1 + b < (a\alpha + b) + \epsilon, \tag{E}$$

$$as_2 + b > (a\beta + b) - \epsilon. \tag{F}$$

Now (C) and (E) imply that $a\alpha + b = \inf T$, while (D) and (F) imply that $a\beta + b = \sup T$.

CASE 3. Suppose that $a < 0$. Multiplying the inequalities in (A) by a shows that

$$a\alpha \ge as \quad \text{and} \quad as \ge a\beta \quad \text{for all} \quad s \in S.$$

Therefore,

$$a\alpha + b \ge as + b \quad \text{for all} \quad s \in S, \tag{G}$$

$$as + b \ge a\beta + b \quad \text{for all} \quad s \in S. \tag{H}$$

Multiplying the inequalities in (B) by a shows that

$$as_1 > a\alpha - \epsilon \quad \text{and} \quad as_2 < a\beta + \epsilon,$$

since $|a| = -a$. Therefore,

$$as_1 + b > (a\alpha + b) - \epsilon, \tag{I}$$

$$as_2 + b < (a\beta + b) + \epsilon. \tag{J}$$

Now (G) and (I) imply that $a\alpha + b = \sup T$, while (H) and (J) imply that $a\beta + b = \inf T$.

1.2 MATHEMATICAL INDUCTION

1.2.1. P_1 is obvious. If P_n is true, then

$$\begin{aligned}1+3+\cdots+(2n+1) &= [1+3+\cdots(2n-1)]+(2n+1)\\ &= n^2+(2n+1) \quad (\text{by } P_n)\\ &= (n+1)^2,\end{aligned}$$

so P_{n+1} is true.

1.2.2. P_1 is obvious. If P_n is true, then

$$\begin{aligned}1^2+2^2+\cdots+(n+1)^2 &= [1^2+2^2+\cdots+n^2]+(n+1)^2\\ &= \frac{n(n+1)(2n+1)}{6}+(n+1)^2 \quad (\text{by } P_n)\\ &= \frac{(n+1)(n+2)(2n+3)}{6},\end{aligned}$$

so P_{n+1} is true.

1.2.3. Since $1=1$, P_1 is true. Suppose that $n \ge 1$ and P_n is true. Then

$$\begin{aligned}1^2+3^2+\cdots+(2n+1)^2 &= \left[1^2+3^2+\cdots+(2n-1)^2\right]+(2n+1)^2\\ &= \frac{n(4n^2-1)}{3}+(2n+1)^2 \quad (\text{by } P_n)\\ &= (2n+1)\left[\frac{n(2n-1)}{3}+2n+1\right]\\ &= \frac{2n+1}{3}[2n^2+5n+3]\\ &= \frac{(n+1)(2n+1)(2n+3)}{3}\\ &= \frac{(n+1)[2(n+1)-1][2(n+1)+1]}{3}\\ &= \frac{(n+1)[4(n+1)^2-1]}{3};\end{aligned}$$

that is,

$$1^2+3^2+\cdots+(2n+1)^2 = \frac{n[4(n+1)^2-1)}{3},$$

which is P_{n+1}.

1.2.4. P_1 is trivial. P_2 is true by the triangle inequality (Theorem 1.1.1). If $n \ge 2$ and P_n is true then

$$\begin{aligned}|a_1+a_2+\cdots+a_n+a_{n+1}| &= |(a_1+a_2+\cdots+a_n)+a_{n+1}|\\ &\le |a_1+a_2+\cdots+a_n|+|a_{n+1}|\end{aligned} \tag{A}$$

by the triangle inequality with $a = a_1 + a_2 + \cdots + a_n$ and $b = a_{n+1}$. By P_n,

$$|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|.$$

This and (A) imply P_{n+1}.

1.2.5. P_1 is obvious. If P_n is true then

$$\begin{aligned}(1 + a_1)(1 + a_2)\cdots(1 + a_{n+1}) &= [(1 + a_1)\cdots(1 + a_n)](1 + a_{n+1})\\ &\ge (1 + a_1 + \cdots + a_n)(1 + a_{n+1}) \quad (\text{by } P_n)\\ &= 1 + a_1 + \cdots + a_{n+1} + a_{n+1}(a_1 + \cdots + a_n),\end{aligned}$$

which implies P_{n+1}, since $a_i \ge 0$.

1.2.6. P_1 is obvious. If P_n is true then

$$\begin{aligned}(1 - a_1)(1 - a_2)\cdots(1 - a_{n+1}) &\ge (1 - a_{n+1})(1 - a_1 - \cdots - a_n) \quad (\text{by } P_n)\\ &= 1 - a_1 - \cdots - a_{n+1} + a_{n+1}(a_1 + \cdots + a_n),\end{aligned}$$

which implies P_{n+1}.

1.2.7. If $s_n > 0$ then $0 < e^{-s_n} < 1$, so $0 < s_{n+1} < 1$. Therefore P_1 is true and P_n implies P_{n+1}.

1.2.8. Subtracting $\sqrt{R} = \frac{1}{2}\left[R\big/\sqrt{R} + \sqrt{R}\right]$ from the equation defining x_{n+1} yields (A) $x_{n+1} - \sqrt{R} = (x_n - \sqrt{R})^2/(2x_n)$. Since $x_0 > 0$, this implies by induction that $x_n > \sqrt{R}$ for all $n \ge 1$, and the definition of x_{n+1} now implies that $x_{n+1} < x_n$ if $n \ge 1$. Now let P_n be the proposition that $x_n - \sqrt{R} \le 2^{-n}(x_0 - \sqrt{R})^2/x_0$. Setting $n = 0$ in (A) verifies P_1. $n = 0$. Since $0 < (x_n - \sqrt{R})/x_n < 1$, (A) implies that $x_{n+1} - \sqrt{R} < (x_n - \sqrt{R})/2$, so P_n implies P_{n+1}.

1.2.9. **(a)** Rewrite the formula as

$$a_{n+1} = \frac{2a_n}{(2n + 1)(2n + 2)}.$$

We compute a few terms to formulate P_n:

$$a_2 = \frac{2a_1}{3 \cdot 4} = \frac{2^2}{4!},$$

$$a_3 = \frac{2a_2}{5 \cdot 6} = \frac{2}{5 \cdot 6}\,\frac{2^2}{4!} = \frac{2^3}{6!},$$

$$a_4 = \frac{2a_3}{7 \cdot 8} = \frac{2}{7 \cdot 8}\,\frac{2^3}{6!} = \frac{2^4}{8!},$$

Let P_n be the proposition that

$$a_n = \frac{2^n}{(2n)!},$$

which is true for $n = 1$ (also for $n = 2, 3, 4$). If P_n is true then

$$a_{n+1} = \frac{2a_n}{(2n+1)(2n+2)} = \frac{2}{(2n+1)(2n+2)} \frac{2^n}{(2n)!} = \frac{2^{n+1}}{(2n+2)!},$$

so P_n implies P_{n+1}.

(b) We compute a few terms to formulate P_n:

$$a_2 = \frac{3a_1}{5} = \frac{3}{4 \cdot 5},$$

$$a_3 = \frac{3a_2}{7 \cdot 6} = \frac{3}{6 \cdot 7} \frac{3}{4 \cdot 5} = \frac{3^2}{4 \cdot 5 \cdot 6 \cdot 7},$$

$$a_4 = \frac{3a_3}{8 \cdot 9} = \frac{3}{8 \cdot 9} \frac{3^2}{4 \cdot 5 \cdot 6 \cdot 7} = \frac{3^3}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}.$$

Multiply numerator and denominator by $3! = 6$ in each fraction yields simpler results:

$$a_2 = \frac{2 \cdot 3^2}{5!}, \quad a_3 = \frac{2 \cdot 3^2}{7!} \quad a_4 = \frac{2 \cdot 3^4}{9!}.$$

Let P_n to be proposition that

$$a_n = \frac{2 \cdot 3^n}{(2n+1)!},$$

which is true for $n = 1$ (also for $n = 2, 3, 4$). If P_n is true then

$$a_{n+1} = \frac{3a_n}{(2n+3)(2n+2)} = \frac{3}{(2n+3)(2n+2)} \frac{2 \cdot 3^n}{(2n+1)!} = \frac{2 \cdot 3^{n+1}}{(2n+3)!},$$

so P_n implies P_{n+1}.

(c) Rewrite the formula as

$$a_{n+1} = \frac{(2n+1)(2n+2)a_n}{2(n+1)^2}.$$

We compute a few terms to formulate P_n:

$$a_2 = \frac{3 \cdot 4}{2(2)^2} a_1 = \frac{3 \cdot 4}{2(2)^2},$$

$$a_3 = \frac{5 \cdot 6}{2(3)^2} a_2 = \frac{5 \cdot 6}{2(3)^2} \frac{3 \cdot 4}{2(2)^2} = \frac{3 \cdot 4 \cdot 5 \cdot 6}{2^2(2 \cdot 3)^2},$$

$$a_4 = \frac{7 \cdot 8}{2(4^2)} a_3 = \frac{7 \cdot 8}{2(4^2)} \frac{3 \cdot 4 \cdot 5 \cdot 6}{2^2(2 \cdot 3)^2} = \frac{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{2^3(2 \cdot 3 \cdot 4)}.$$

Let P_n be the proposition that

$$a_n = \frac{(2n)!}{2^n (n!)^2},$$

which is true for $n = 1$ (also for $n = 2, 3, 4$). If P_n is true then

$$a_{n+1} = \frac{(2n+1)(2n+2)}{2(n+1)^2}a_n = \frac{(2n+1)(2n+2)}{2(n+1)^2}\,\frac{(2n)!}{2^n(n!)^2} = \frac{(2n+2)!}{2^{n+1}(n+1)^2},$$

so P_n implies P_{n+1}.

(d) Rewrite the formula as

$$a_{n+1} = \left(\frac{n+1}{n}\right)^n.$$

We compute a few terms to formulate P_n:

$$a_2 = 2a_1 = 2,$$

$$a_3 = \left(\frac{3}{2}\right)^2 a_2 = \left(\frac{3}{2}\right)^2 2 = \frac{3^2}{2},$$

$$a_4 = \left(\frac{4}{3}\right)^3 a_3 = \left(\frac{4}{3}\right)^3 \frac{3^2}{2} = \frac{4^3}{3\cdot 2}.$$

We take P_n to be the proposition that

$$a_n = \frac{n^{n-1}}{(n-1)!},$$

which is true for $n = 1$ (also for $n = 2, 3, 4$). If P_n is true, then

$$a_{n+1} = \left(\frac{n+1}{n}\right)^n a_n = \left(\frac{n+1}{n}\right)^n \frac{n^{n-1}}{(n-1)!} = \frac{(n+1)^n}{n!},$$

so P_n implies P_{n+1}.

1.2.10. If $a_n = n!$ then $a_{n+1} = (n+1)a_n = (n+1)n! = (n+1)!$, so P_n implies P_{n+1}. However, P_n is false for all n, since $a_n = 0$ for all n, by induction.

1.2.11. **(a)** If P_n is true then

$$\begin{aligned} 1+2+\cdots+n+n+1 &= \frac{(n+2)(n-1)}{2} + (n+1) \\ &= \frac{n^2+3n}{2} = \frac{(n+3)n}{2}, \end{aligned}$$

so P_n implies P_{n+1}.

(b) No; see Example 1.2.1.

1.2.12. Calculation shows that the first integer for which the inequality is true is $n_0 = 6$. If $n \ge 6$ and $1/n! > 8^n/(2n)!$ then

$$\frac{1}{(n+1)!} = \frac{1}{n!}\frac{1}{n+1} > \frac{8^n}{(2n)!}\frac{1}{n+1}. \tag{A}$$

But

$$\frac{1}{n+1} - \frac{8}{(2n+2)(2n+1)} = \frac{2n-3}{(n+1)(2n+1)} > 0, \quad n \ge 2.$$

This and (A) imply that

$$\frac{1}{(n+1)!} > \frac{8^n}{(2n)!}\frac{8}{(2n+2)(2n+1)} = \frac{8^{n+1}}{(2n+2)!},$$

so P_n implies P_{n+1}.

1.2.13. **(a)** If $n \ge 0$, let P_n be the proposition that $n = aq + r$ where q is an integer and $0 \le r < a$. Then P_n is true if $0 \le n < a$, since $n = a \cdot 0 + n$. P_a is true, since $a = a \cdot 1 + 0$ Now suppose that $n \ge a$ and $P_0, P_1, \dots, P_n$ are true. Then $n + 1 - a \le n$, so $n + 1 - a = q_0 a + r$ with $0 \le r < a$, by the induction assumption. Hence $n + 1 = (q_0 + 1)a + r$. This implies P_{n+1}.

(b) If $b < 0$ then $0 \le b + |b|a = aq_1 + r$ with $0 \le r < a$, from **(a)**; hence $b = aq + r$ with $q = q_1 - |b|$.

(c) Suppose that $aq + r = aq_1 + r_1$, where q, q_1, r and r_1 are integers with (A) $0 \le r, r_1 < a$. Then $a|q - q_1| = |r_1 - r|$. If $q \ne q_1$ then $a|q - q_1| \ge a$, which implies that $|r_1 - r| \ge a$, contradicting (A). Therefore $q_1 = q$ and $r_1 = r$.

1.2.14. **(a)** P_1 is true, from the definition in Example 1.2.7. Now assume that P_k is true for some $k \ge 1$. Suppose that that p divides $p_1 p_2 \cdots p_k p_{k+1} = ab$, with $a = p_1 \cdots p_k$ and $b = p_{k+1}$. From the given statement, p divides a or b. If p divides a then $p = p_i$ for some i in $\{1, \dots, k\}$, by P_k. If p divides b then $p = p_{k+1}$, by P_1. Therefore P_k implies P_{k+1}.

(b) Let P_k be the proposition that the assertion is true if $\min(r, s) = k$. Then P_1 is true, by the definition of prime in Example 1.2.7. Assume that P_k is true for some $k \ge 1$. Now suppose that $p_1 \cdots p_k p_{k+1} = q_1 \cdots q_s$, where $p_1, \dots, p_k, p_{k+1}$ and $q_1, \dots, q_s$ are positive primes and $s \ge k + 1$. Since p_{k+1} divides $q_1 \cdots q_s$, **(a)** implies that $p_{k+1} = q_i$ for some i in $\{1, \dots, s\}$. We may assume that $p_{r+1} = q_s$. Then $p_1 \cdots p_k = q_1 \cdots q_{s-1}$, and P_k implies that $k = s - 1$ and $\{p_1, \cdots, p_k\}$ is a permutation of $\{q_1, \cdots, q_k\}$. This implies P_{k+1}.

1.2.15. P_1 and P_2 are true by inspection. Suppose that P_n and P_{n-1} are true for some $n \ge 2$. Then

$$\begin{aligned} a_{n+1} &= [3^n - (-2)^n] + 6\left[3^{n-1} - (-2)^{n-1}\right] \\ &= [3^n + 6 \cdot 3^{n-1}] - [(-2)^n + 6(-2)^{n-1}] \\ &= [3^n + 2 \cdot 3^n] - [(-2)^n - 3(-2)^n] \\ &= 3^{n+1} - (-2)^{n+1}, \end{aligned}$$

which implies P_{n+1}. Apply Theorem 1.2.3.

1.2.16. P_1, P_2 and P_3 are true by inspection. Suppose that P_n, P_{n-1} and P_{n-2} are true for some $n \geq 3$. Then

$$\begin{aligned} a_{n+1} &= 9(3^{n-1} - 5^{n-1} + 2) - 23(3^{n-2} - 5^{n-2} + 2) + 15(3^{n-3} - 5^{n-3} + 2) \\ &= (9 \cdot 3^{n-1} - 23 \cdot 3^{n-2} + 15 \cdot 3^{n-3}) \\ &\quad -(9 \cdot 5^{n-1} - 23 \cdot 5^{n-2} + 15 \cdot 5^{n-3}) + (9 - 23 + 15)2 \\ &= (27 - 23 + 5)3^{n-2} - (45 - 23 + 3)5^{n-2} + 2 = 3^n - 5^n + 2, \end{aligned}$$

which implies P_{n+1}. Use Theorem 1.2.3.

1.2.17. Routine computations verify P_1 and P_2. Now suppose that $n \geq 2$ and assume that

$$F_k = \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}} \tag{A}$$

for $1 \leq k \leq n$. Then

$$\begin{aligned} F_{n+1} = F_n + F_{n-1} &= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} + \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left[\frac{1+\sqrt{5}}{2} + 1\right] - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left[\frac{1-\sqrt{5}}{2} + 1\right] \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left[\frac{3+\sqrt{5}}{2}\right] - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left[\frac{3-\sqrt{5}}{2}\right] \\ &= \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}, \end{aligned}$$

since

$$\frac{3 \pm \sqrt{5}}{2} = \frac{(1 \pm \sqrt{5})^2}{4}.$$

1.2.18. Let P_n be the proposition that the statement is true for all $r > -1$. Since $\int_0^1 (1-y)^r\,dy = \frac{1}{r+1}$, P_0 is true. Now suppose that $n \geq 0$ and P_n is true. Integration by parts yields

$$\begin{aligned} \int_0^1 y^{n+1}(1-y)^r\,dy &= -\frac{y^{n+1}}{(1-y)^{r+1}}r+1\Big|_0^1 + \frac{n+1}{r+1}\int_0^1 y^n(1-y)^{r+1}\,dy \\ &= \frac{n+1}{r+1}\int_0^1 y^n(1-y)^{r+1}\,dy. \end{aligned} \tag{A}$$

P_n implies that $\int_0^1 y^n(1-y)^{r+1}\,dy = \frac{n!}{(r+2)(r+3)\cdots(r+n+2)}$ if $r > -2$. This and (A) imply that $\int_0^1 y^{n+1}(1-y)^r\,dr = \frac{(n+1)!}{(r+1)(r+2)\cdots(r+n+2)}$, verifying P_{n+1}.

1.2.19.

$$\textbf{(a)} \sum_{m=0}^{n+1}\binom{n+1}{m}t^m = (1+t)^{n+1} = (1+t)(1+t)^n = (1+t)\sum_{m=0}^{n}\binom{n}{m}t^m$$
$$= \sum_{m=0}^{n}\binom{n}{m}t^m + \sum_{m=0}^{n}\binom{n}{m}t^{m+1}$$
$$= \sum_{m=0}^{n}\binom{n}{m}t^m + \sum_{m=1}^{n+1}\binom{n}{m-1}t^m$$
$$= \sum_{m=0}^{n+1}\left[\binom{n}{m} + \binom{n}{m-1}\right]t^m,$$

since $\binom{n}{-1} = \binom{n}{n+1} = 0$. Comparing coefficients of t^m in the first and last expressions shows that

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}. \tag{A}$$

Now let P_n be the proposition that

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}, \quad 0 \le m \le n.$$

Then P_0 is true, since $(1+t)^0 = 1$. Now suppose that P_n is true for some $n \ge 0$. If $1 \le m \le n$, then (A) and P_n imply that

$$\binom{n+1}{m} = \frac{n!}{m!(n-m)!} + \frac{n!}{(m-1)!(n-m+1)!}$$
$$= \frac{n!}{(m-1)!(n-m)!}\left[\frac{1}{m} + \frac{1}{n-m+1}\right]$$
$$= \frac{n!}{(m-1)!(n-m)!}\,\frac{n+1}{m(n-m+1)} = \frac{(n+1)!}{m!(n-m+1)!}.$$

This verifies the assertions in P_{n+1} for $1 \le m \le n$, and we have seen they hold automatically for $m = 0$ and $m = n+1$ (that is, $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$). Therefore P_n implies P_{n+1}.

(b) Set $t = -1$ in the identity $(1+t)^n = \sum_{m=0}^{n}\binom{n}{m}t^n$ to obtain the first sum, $t = 1$ to obtain the second.

(c) $(x+y)^n = y^n(1+x/y)^n$. From **(a)** with $t = x/y$, we see from this that $(x+y)^n = y^n \sum_{m=0}^{n} \binom{n}{m} (x/y)^m = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}$.

1.2.20. We take constants of integration to be zero. Integrating by parts yields the first antiderivative

$$A_1(x) = \int \log x \, dx = x \log x - \int 1 \, dx = x \log x - x.$$

Integration by parts yields the second antiderivative

$$A_2(x) = \int (x \log x - x) \, dx = \frac{x^2}{2} \log x - \int \left(\frac{x}{2} + x \right) \, dx = \frac{x^2}{2} - \frac{3x^2}{4}.$$

We conjecture that an n-th antiderivative is

$$A_n(x) = \frac{x^n}{n!} \left(\log x - k_n \right), \tag{A}$$

for a suitable constant k_n to be determined later.

We have already verified (A) for $n = 1$ and 2, with $k_1 = 1$ and $k_2 = 3/2$. If we assume that (A) is true for a given n then

$$A_{n+1} = \frac{x^{n+1}}{(n+1)!} \log x - \frac{1}{n!} \int \left(\frac{x^n}{n+1} + k_n x^n \right) \, dx = \frac{x^{n+1}}{(n+1)!} \left(\log x + k_{n+1} \right),$$

where

$$k_{n+1} = k_n + \frac{1}{n+1}.$$

Now an easy induction yields $k_n = \sum_{j=1}^{n} 1/j$. Therefore

$$A_n = \frac{x^n}{n!} \left(\ln x - \sum_{j=1}^{n} \frac{1}{j} \right)$$

is an n-th antiderivative of $\log x$ for every n.

1.2.21. $f_2(x_1, x_2) = x_1 + x_2 + |x_1 - x_2| = 2 \max(x_1, x_2)$ from Exercise 1.1.1**(a)** with $a = x_1$ and $b = x_2$. Now

$$\begin{aligned} f_3(x_1, x_2, x_3) &= f_2(x_1, x_2) + 2x_3 + |f_2(x_1, x_2) - 2x_3| \\ &= 2 \left[\max(x_1, x_2) + x_3 + |\max(x_1, x_2) - x_3| \right] \\ &= 4 \max\left(\max(x_1, x_2), x_3 \right) \end{aligned}$$

from Exercise 1.1.1**(a)** with $a = \max(x_1, x_2)$ and and $b = x_3$. Since

$$\max\left(\max(x_1, x_2), x_3 \right) = \max(x_1, x_2, x_3), \quad f_3(x_1, x_2, x_3) = 4 \max(x_1, x_2, x_3).$$

Now let P_n be the proposition that

$$f_n(x_1, x_2, \dots, x_n) = 2^{n-1}\max(x_1, x_2, \dots, x_n).$$

If P_n is true, then

$$\begin{aligned} f_{n+1}(x_1, x_2, \dots, x_{n+1}) = {} & f_n(x_1, x_2, \dots, x_n) + 2^{n-1}x_{n+1} \\ & + \left| f_n(x_1, x_2, \dots, x_n) - 2^{n-1}x_{n+1} \right| \\ = {} & 2^{n-1}\left[\max(x_1, x_2, \dots, x_n) + x_{n+1} \right. \\ & \left. + |\max(x_1, x_2, \dots, x_n) - x_{n+1}|\right] \\ = {} & 2^{n-1}\max\left(\max(x_1, x_2, \dots, x_n), x_{n+1}\right) \end{aligned}$$

from Exercise 1.1.1**(a)** with $a = \max(x_1, x_2, \dots, x_n)$ and $b = x_{n+1}$. Since

$$\max\left(\max(x_1, x_2, \dots, x_n), x_{n+1}\right) = \max(x_1, x_2, \dots, x_{n+1})$$

this implies P_{n+1}. A similar argument using Exercise 1.1.1**(b)** shows that $g_n(x_1, x_2, \dots, x_n) = 2^{n-1}\min(x_1, x_2, \dots, x_n)$.

1.2.22. P_1 is the assertion that

$$\sin x = \frac{1 - \cos 2x}{2\sin x}.$$

To verify this we let $A = B = x$ in the second given identity, to see that

$$\cos 2x = \cos^2 x - \sin^2 x.$$

Therefore

$$\begin{aligned} \frac{1 - \cos 2x}{2\sin x} &= \frac{1 - \cos^2 x + \sin^2}{2\sin x} \\ &= \frac{2\sin^2 x}{2\sin x} = \sin x. \end{aligned}$$

Now suppose that $n \ge 1$ and P_n is true. Then

$$\begin{aligned} \sin x + \sin 3x + \dots + \sin(2n+1)x &= [\sin x + \sin 3x + \dots + \sin(2n-1)x] + \sin(2n+1)x \\ &= \frac{1 - \cos 2nx}{2\sin x} + \sin(2n+1)x \quad (\text{from } P_n) \\ &= \frac{1 - \cos 2nx + 2\sin x\sin(2n+1)x}{2\sin x}; \end{aligned}$$

that is,

$$\sin x + \sin 3x + \dots + \sin(2n+1)x = \frac{1 - \cos 2nx + 2\sin x\sin(2n+1)x}{2\sin x}. \qquad \text{(A)}$$

To handle the product of sines in the numerator let $A = (2n+1)x$ and $B = x$ in the given identities. This yields

$$\begin{aligned}\cos 2nx &= \cos(2n+1)x\cos x + \sin(2n+1)x\sin x\\ \cos(2n+2)x &= \cos(2n+1)x\cos x - \sin(2n+1)x\sin x.\end{aligned}$$

Subtracting the first of these identities from the second yields

$$2\sin(2n+1)x\sin x = \cos 2nx - \cos(2n+2)x,$$

and substituting this into (A) yields

$$\sin x + \sin 3x + \cdots + \sin(2n+1)x = \frac{1-\cos 2nx + \cos 2nx - \cos(2n+2)x}{2\sin x},$$

so

$$\sin x + \sin 3x + \cdots + \sin(2n+1)x = \frac{1-\cos(2n+2)x}{2\sin x},$$

which is P_{n+1}.

1.2.23. Let P_n be the stated proposition. P_1 is trivial. Suppose that $n > 1$ and P_{n-1} is true. If $\ell_n = n$, P_{n-1} implies P_n. If $\ell_n = s < n$, choose r so that $\ell_r = n$, and define

$$\ell_i' = \begin{cases}\ell_i & \text{if } i \ne r \text{ and } i \ne n,\\ s & \text{if } i = r,\\ n & \text{if } i = n.\end{cases}$$

Then

$$\begin{aligned}Q(\ell_1,\ell_2,\dots,\ell_n) - Q(\ell_1',\ell_2',\dots,\ell_n') &= (x_n-y_s)^2 + (x_r-y_n)^2\\ &\quad - (x_n-y_n)^2 - (x_r-y_s)^2\\ &= 2(x_n-x_r)(y_n-y_s) \ge 0. \qquad \text{(A)}\end{aligned}$$

Since $\ell_n' = n$, P_{n-1} implies that

$$Q(\ell_1',\ell_2',\dots,\ell_n') \ge Q(1,2,\dots,n).$$

This and (A) imply P_n.

1.3 THE REAL LINE

1.3.3. Suppose that $a \in A$. We consider two cases: **(i)** Suppose that $a \in X$. Then $a \in A \cap X$ and **(b)** implies that $a \in B \cap X$, which implies that $a \in B$. **(ii)** Suppose that $a \notin X$. Since $a \in A \cup X$, **(a)** implies that $a \in B \cup X$. Therefore, $a \in B$.

Since $a \in X$ and $a \notin X$ are the only two possibilities, it follows that $a \in B$. Therefore, $A \subseteq B$. Similarly, $B \subseteq A$, so $A = B$.

1.3.6. $x \in (S \cap T)^c \Leftrightarrow x \in S^c$ or $x \in T^c \Leftrightarrow x \in S^c \cup T^c$.

$x \in (S \cup T)^c \Leftrightarrow x \in S^c$ and $x \in T^c \Leftrightarrow x \in S^c \cap T^c$.

1.3.7. **(a)** $x \in I^c \Leftrightarrow x \in F^c$ for some F in $F \Leftrightarrow x \in \cup\{F^c \mid F \in F\}$.

(b) $x \in U^c \Leftrightarrow x \in F^c$ for every F in $F \Leftrightarrow x \in \cap\{F^c \mid F \in F\}$.

1.3.8. **(a)** If $S_1, \dots, S_n$ are open and x is in $\cap_{i=1}^n S_i$, there are positive numbers $\epsilon_1, \dots, \epsilon_n$ such that $(x-\epsilon_i, x+\epsilon_i) \subset S_i$. If $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$, then $(x-\epsilon, x+\epsilon) \subset \bigcap_{i=1}^n S_i$.

(b) $\bigcap_{n=1}^\infty (1/n, -1/n) = \{0\}$.

1.3.9. **(a)** Let $T = \bigcup_{i=1}^n T_i$, where $T_1, \dots, T_n$ are closed. Then $T^c = \bigcap_{i=1}^n T_i^c$ (Exercise 1.3.7). Since T_i^c is open, so is T^c (Exercise 1.3.8). Hence, T is closed.

(b) $\bigcup_{n=1}^\infty [1/n, \infty) = (0, \infty)$.

1.3.10. **(a)** Since U is a neighborhood of x_0, there is an $\epsilon > 0$ such that $(x_0-\epsilon, x_0+\epsilon) \subset U$. Since $U \subset V$, $(x_0-\epsilon, x_0+\epsilon) \subset V$. Hence, V is a neighborhood of x_0.

(b) Since $U_1, U_2, \dots, U_n$ are neighborhoods of x_0, there are positive numbers $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ such that $(x_0-\epsilon_i, x_0+\epsilon_i) \subset U_i$ $(1 \le i \le n)$. If $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$, then $(x_0-\epsilon, x_0+\epsilon) \subset \bigcap_{i=1}^n U_i$. Hence, $\bigcap_{i=1}^n U_i$ is a neighborhood of x_0.

1.3.12. If x_0 is a limit point of S, then every neighborhood of x_0 contains points of S other than x_0. If every neighborhood of x_0 also contains a point in S^c, then $x_0 \in \partial S$. If there is a neighborhood of x_0 that does not contain a point in S^c, then $x_0 \in S^0$. These are tho only possibilities.

1.3.13. An isolated point x_0 of S has a neighborhood V that contains no other points of S. Any neighborhood U of x_0 contains $V \cap U$, also a neighborhood of x_0 (Exercise 1.3.10**(b)**), so $S^c \cap U \ne \emptyset$. Since $x_0 \in S \cap U$, $x_0 \in \partial S$.

1.3.14. **(a)** If $x_0 \in \partial S$ and U is a neighborhood of x_0 then, (A) $U \cap S \ne \emptyset$. If x_0 is not a limit point of S, then (B) $U \cap (S - \{x_0\}) = \emptyset$ for some U. Now (A) and (B) imply that $x_0 \in S$, and (B) implies that x_0 is an isolated point of S.

(b) If S is closed, Corollary 1.3.6 and **(a)** imply that $\partial S \subset S$; hence, $\overline{S} = S \cup \partial S = S$. If $\overline{S} = S$, then $\partial S \subset S$. Since $S^0 \subset S$, S is closed, by Exercise 1.3.12 and Corollary 1.3.6.

1.3.15. False: 0 is a limit point of $\left\{\dfrac{1}{n} \,\middle|\, n = 1, 2, \dots\right\}$, which consists entirely of isolated points.

1.3.16. **(a)** If $\epsilon > 0$, then $S \cap (\beta-\epsilon, \beta) \ne \emptyset$ and $S^c \cap (\beta, \beta+\epsilon) \ne \emptyset$. **(b)** If S is bounded below and $\alpha = \inf S$ then $\alpha \in \partial S$.

1.3.17. $\inf S$ and $\sup S$ are in ∂S (Exercise 1.3.16) and $\partial S \subset S$ if S is closed (Exercise 1.3.14).

1.3.18. Suppose that $a \in S$ and $H = \{r \mid r > 0 \text{ and } (a-r, a+r) \subset S\}$. Since S is open, $H \ne \emptyset$. If $S \ne \mathbb{R}$ then H is bounded. Let $\rho = \sup H$. Then $a-\rho$ and $a+\rho$ are limit points of S and therefore in S, which is closed. Since S is open, there is an $\epsilon > 0$ such that $(a-\rho-\epsilon, a-\rho+\epsilon)$ and $(a+\rho-\epsilon, a+\rho+\epsilon)$ are in S. Since $(a-\rho+\epsilon/2, a+\rho-\epsilon/2) \subset S$ it follows that $(a-\rho-\epsilon, a+\rho+\epsilon) \subset S$, which contradicts the definition of ρ. Hence, $S = \mathbb{R}$.

1.3.19. **(a)** If x_0 is a limit point of ∂S and $\epsilon > 0$, there is an x_1 in $(x_0-\epsilon, x_0+\epsilon) \cap \partial S$. Since $(x_0-\epsilon, x_0+\epsilon)$ is a neighborhood of x_1 and $x_1 \in \partial S$, $(x_0-\epsilon, x_0+\epsilon) \cap S \ne \emptyset$ and

$(x_0 - \epsilon, x_0 + \epsilon) \cap S^c \neq \emptyset$. Therefore, $x_0 \in \partial S$ and ∂S is closed (Corollary 1.3.6).

(b) If $x_0 \in S^0$ then $(x_0 - \epsilon, x_0 + \epsilon) \subset S$ for some $\epsilon > 0$. Since $(x_0 - \epsilon, x_0 + \epsilon) \subset S^0$ (Example 1.3.4), S^0 is open.

(c) Apply **(b)** to S^c.

(d) If x_0 is not a limit point of S, there is a neighborhood of x_0 that contains no points of S distinct from x_0. Therefore, the set of points that are not limit points of S is open, and the set of limit points of S is consequently closed.

(e) $\left(\overline{S}\right)^c =$ exterior of S, which is open, by **(b)**. Hence, $\overline{S}$ is closed and $\overline{\left(\overline{S}\right)} = \overline{S}$ from (Exercise 1.3.14**(b)**), applied to $\overline{S}$.

1.3.19. **(a)** If x_0 is a limit point of ∂S and $\epsilon > 0$, there is an x_1 in $(x_0 - \epsilon, x_0 + \epsilon) \cap \partial S$. Since $(x_0 - \epsilon, x_0 + \epsilon)$ is a neighborhood of x_1 and $x_1 \in \partial S$, $(x_0 - \epsilon, x_0 + \epsilon) \cap S \neq \emptyset$ and $(x_0 - \epsilon, x_0 + \epsilon) \cap S^c \neq \emptyset$. Therefore, x_0 is in ∂S and ∂S is closed (Corollary 1.3.6).

(b) If $x_0 \in S^0$, then $(x_0 - \epsilon, x_0 + \epsilon) \subset S$ for some $\epsilon > 0$. Since $(x_0 - \epsilon, x_0 + \epsilon) \subset S^0$ (Example 1.3.4), S^0 is open.

(c) Apply **(b)** to S^c.

(d) If x_0 is not a limit point of S, there is a neighborhood of x_0 that contains no points of S distinct from x_0. Therefore, the set of points that are not limit points of S is open, and the set of limit points of S is consequently closed.

(e) $\left(\overline{S}\right)^c =$ exterior of S, which is open, by **(b)**. Hence, $\overline{S}$ is closed and $\overline{\left(\overline{S}\right)} = \overline{S}$ from (Exercise 1.3.14**(b)**), applied to $\overline{S}$.

1.3.21. Since $H_1 = \left\{(s-1, s+1) \,\middle|\, s \in S\right\}$ is an open covering for S, $S \subset \bigcup_{j=1}^{n}(s_j - 1, s_j + 1)$ for some $s_1 < s_2 < \cdots < s_n$ in S. Therefore, $S \subset (s_1 - 1, s_n + 1)$, and S is bounded. If S is not closed, it has a limit point $x_0 \notin S$ (Theorem 1.3.5). Then $H_2 = \left\{(s - |s - x_0|/2, s + |s - x_0|/2) \,\middle|\, s \in S\right\}$ is an open covering for S, but if s_1, s_2, $\dots$, s_n are in S and $2\delta = \min\left\{|s_i - x_0| \,\middle|\, 1 \le i \le n\right\}$, then $\bigcup_{i=1}^{n}(s_i - |x_0 - s_i|/2, s_i + |x_0 - s_i|/2)$ does not intersect $(x_0 - \delta, x_0 + \delta)$, which contains points of S. Therefore, no finite subcollection of H_2 covers S.

1.3.22. **(a)** If $t_0 \in T$ and every neighborhood of t_0 contains a point of S, then either $t_0 \in S$ or t_0 is a limit point of S. In either case $t_0 \in \overline{S}$. Consequently, $T \subset \overline{S}$. Conversely, if $T \subset \overline{S} = S \cup \partial S$, any t_0 in T is in S or is a limit point of S (Exercise 1.3.14**(a)**); in either case, every neighborhood of t_0 intersects S.

(b) **(a)** with $T = \mathbb{R}$ (reals) implies that S is dense in $\mathbb{R}$ if and only if every real number is in S or is a limit point of S. This is equivalent to saying that every interval contains a point of S.

1.3.23. **(a)** $x \in (S_1 \cap S_2)^0 \Leftrightarrow x$ has a neighborhood $N \subset S_1 \cap S_2 \Leftrightarrow x \in S_1^0$ and $x \in S_2^0 \Leftrightarrow x \in S_1^0 \cap S_2^0$. **(b)** $x \in S_1^0 \cup S_2^0 \Rightarrow x \in S_1^0$ or $x \in S_2^0 \Rightarrow x$ has a neighborhood N such that $N \subset S_1$ or $N \subset S_2 \Rightarrow x$ has a neighborhood $N \subset S_1 \cup S_2 \Rightarrow x \in (S_1 \cup S_2)^0$.

1.3.24. **(a)** $x \in \partial(S_1 \cup S_2) \Rightarrow$ every neighborhood of x contains a point in $(S_1 \cup S_2)^c$ and a point in $S_1 \cup S_2$. If every neighborhood of x contains points in $S_1 \cap S_2$, then $x \in \partial S_1 \cap \partial S_2 \subset \partial S_1 \cup \partial S_2$. Now suppose that x has a neighborhood N such that

$N \cap S_1 = \emptyset$. If U is any neighborhood of x, then so is $N \cap U$, and $N \cap U \cap S_2 \neq \emptyset$, since $N \cap U$ must intersect $S_1 \cup S_2$. This means that $x \in \partial S_2 \subset \partial S_1 \cup \partial S_2$. A similar argument applies if x has a neighborhood N such that $N \cap S_2 = \emptyset$.

(b) $x \in \partial(S_1 \cap S_2) \Rightarrow$ every neighborhood of x contains a point in $(S_1 \cap S_2)^c$ and a point in $S_1 \cap S_2$. If every neighborhood of x contains a point in $(S_1 \cup S_2)^c$, then $x \in \partial S_1 \cap \partial S_2 \subset \partial S_1 \cup \partial S_2$. Now suppose that x has a neighborhood N such that $N \subset S_1$. If U is any neighborhood of x, then so is $N \cap U$, and $N \cap U \cap S_2^c \neq \emptyset$, since $N \cap U$ must intersect $(S_1 \cap S_2)^c$. This means that $x \in \partial S_2 \subset \partial S_1 \cup \partial S_2$. A similar argument applies if x has a neighborhood N such that $N \subset S_2$.

(c) If $x \in \partial\overline{S}$, then any neighborhood N of x contains points x_0 in $\overline{S}$ and x_1 not in $\overline{S}$. Either $x_0 \in S$ or $x_0 \in \partial S$. In either case $N \cap S \neq \emptyset$. Since $x_1 \in N \cap S^c$, it follows that $x \in \partial S$; hence, $\partial\overline{S} \subset \partial S$.

(d) Obvious from the definition of ∂S.

(e)

$$\begin{aligned} \partial(S - T) &= \partial(S \cap T^c) && \text{(definition of } S - T) \\ &\subset \partial S \cup \partial T^c && \text{(Exercise 1.3,24(\textbf{b}))} \\ &= \partial S \cup \partial T && \text{(Exercise 1.3.24(\textbf{d})).} \end{aligned}$$

CHAPTER 2

Differential Calculus of Functions of One Variable

2.1 FUNCTIONS AND LIMITS

2.1.1. **(a)** If $|x| \le 1$, infinitely many y's satisfy $\sin y = x$; if $|x| > 1$, no y satisfies $\sin y = x$.

(b) $e^y > 0$ for all y, while $-|x| \le 0$ for all x.

(c) $1 + x^2 + y^2 > 0$ for all x and y.

(d) $y(y-1) = x^2$ has two solutions for every x.

2.1.4. **(a)** $\lim_{x\to 1} x^2 + 2x + 1 = 4$;

$$\begin{aligned}|x^2 + 2x + 1 - 4| &= |x^2 + 2x - 3| = |x-1||x+3| \\ &\le |x-1|(|x-1|+4) \le \delta(\delta+4)\end{aligned}$$

if $|x-1| < \delta$. Given $\epsilon > 0$ choose $\delta \le \min(1, \epsilon/5)$. Then

$$|x^2 + 2x - 4| < (\epsilon/5)(1+4) = \epsilon$$

if $|x-1| < \delta$.

(b) $\lim_{x\to 2} \dfrac{x^3 - 8}{x-2} = 10$. **Proof**

$$\begin{aligned}\left|\frac{x^2-8}{x-2} - 10\right| &= |x^2 + 2x + 4 - 10| = |x^2 + 2x - 6| \\ &= |x-2||x+4| \le |x-2|(|x-2|+6) < \delta(\delta+6)\end{aligned}$$

if $|x-2| < \delta$. Given $\epsilon > 0$ choose $\delta \le \min(1, \epsilon/7)$. Then

$$\left|\frac{x^2-8}{x-2} - 10\right| < (\epsilon/7)(1+6) = \epsilon \quad \text{if} \quad |x-1| < \delta.$$

(c) $\lim_{x\to 0} \frac{1}{x^2-1} = -1$. **Proof** $\left|\frac{1}{x^2-1}+1\right| = \left|\frac{x^2}{x^2-1}\right|$. If $|x| < 1/\sqrt{2}$ then $1-x^2 > 1/2$, so $\left|\frac{1}{x^2-1}+1\right| < 2x^2$. Therefore $\left|\frac{1}{x^2-1}+1\right| < \epsilon$ if $|x| < \delta = \min(1/\sqrt{2}, \sqrt{\epsilon/2})$.

(d) $\lim_{x\to 4} \sqrt{x} = 2$. **Proof** $|\sqrt{x}-2| = \left|\frac{x-4}{\sqrt{x}+2}\right| < \frac{|x-4|}{2}$, so $|\sqrt{x}-2| < \epsilon$ if $|x-4| < \delta = 2\epsilon$.

(e) Since $x^3-1 = x^2+x+1$,

$$f(x) = \frac{x^2+x+1}{x-2} + x = \frac{x^2+x+1+x(x-2)}{x-2} = \frac{2x^2-x+1}{x-2} \quad \text{if} \quad x \ne 1.$$

We show that $\lim_{x\to 1} f(x) = -2$.

$$f(x)+2 = \frac{2x^2-x+1}{x-2}+2 = \frac{2x^2-x+1+2(x-2)}{x-2} = \frac{2x^2+x-3}{x-2} = \frac{(x-1)(2x+3)}{x-2}$$

if $x \ne 2$. Therefore

$$|f(x)+2| = |x-1|\left|\frac{2x+3}{x-2}\right| \quad \text{if} \quad x \ne 1. \tag{A}$$

To handle the multiplier

$$\frac{2x+3}{x-2} = \frac{2(x-2)+7}{x-2} = 2 + \frac{7}{x-2},$$

we first restrict x to the interval $(1/2, 3/2)$ (so $|x-1| < 1/2$). On this interval

$$-12 < 2 + \frac{7}{x-2} < -\frac{8}{3}, \qquad \left|\frac{2x+3}{x-2}\right| < 12.$$

From this and (A), $|f(x)+2| < 12|x-1|$ if $0 < |x-1| < 1/2$. If $\epsilon > 0$ is given, let $\delta = \min(\epsilon/12, 1/2)$. Then $|f(x)+2| < \epsilon$ if $0 < |x-1| < \delta$.

2.1.5. If $\lim_{x\to x_0} f(x) = L$ according to Definition 2.1.2 and $\epsilon' > 0$, there is a $\delta > 0$ such that $|f(x)-L| < \epsilon'$ if $0 < |x-x_0| < \delta$. Let $\epsilon' = K\epsilon$ to see that $\lim_{x\to x_0} f(x) = L$ according to the modified definition. If $\lim_{x\to x_0} f(x) = L$ according to the modified definition and $\epsilon' > 0$, there is a $\delta > 0$ such that $|f(x)-L| < K\epsilon'$ if $0 < |x-x_0| < \delta$. Let $\epsilon' = \epsilon/K$ to see that $\lim_{x\to x_0} f(x) = L$ according to Definition 2.1.2.

2.1.7. **(a)** If $x \neq 0$ then $\dfrac{x+|x|}{x} = \begin{cases} 0 \text{ if } x < 0, \\ 2 \text{ if } x > 0. \end{cases}$ If ϵ and δ are arbitrary positive numbers, then $\left|\dfrac{x+|x|}{x} - 0\right| = 0 < \epsilon$ if $-\delta < x < 0$ and $\left|\dfrac{x+|x|}{x} - 2\right| = 0 < \epsilon$ if $0 < x < \delta$. Therefore $\lim_{x\to 0-} \dfrac{x+|x|}{x} = 0$ and $\lim_{x\to 0+} \dfrac{x+|x|}{x} = 2$.

(b) Let $f(x) = x\cos\dfrac{1}{x} + \sin\dfrac{1}{x} + \sin\dfrac{1}{|x|}$. We first observe that $\lim_{x\to 0} x\cos\dfrac{1}{x} = 0$, since if $|x| < \delta \le \epsilon$ then $\left|x\cos\dfrac{1}{x}\right| \le |x| < \epsilon$. If $x < 0$ then $f(x) = x\cos\dfrac{1}{x}$ (since $\sin u = -\sin(-u)$), so $\lim_{x\to 0-} f(x) = 0$. If $x > 0$ then $f(x) = x\cos\dfrac{1}{x} + 2\sin\dfrac{1}{x}$. Therefore $\lim_{x\to 0+} f(x) = 0$ does not exist, because if it did we would have $\lim_{x\to 0+} 2\sin\dfrac{1}{x} = \lim_{x\to 0+} f(x) - \lim_{x\to 0} x\cos\dfrac{1}{x} = \lim_{x\to 0+} f(x) - 0 = \lim_{x\to 0+} f(x)$, which is impossible since $2\sin\dfrac{1}{x}$ oscillates between ± 2 as $x \to 0+$.

(c)

$$\frac{|x-1|}{x^2+x-2} = \frac{|x-1|}{(x+2)(x-1)} = \begin{cases} \dfrac{1}{x+2}, & x > 1 \\ -\dfrac{1}{x+2}, & x < 1. \end{cases} \tag{A}$$

$\lim_{x\to 1} \dfrac{1}{x+2} = \dfrac{1}{3}$, since $\left|\dfrac{1}{x+2} - \dfrac{1}{3}\right| = \left|\dfrac{x-1}{3(x+2)}\right| < \dfrac{|x-1|}{6} < \epsilon$ if $|x-1| < \min(1, 6\epsilon)$. This and (A) imply that

$$\lim_{x\to 1+} \frac{|x-1|}{x^2+x-2} = \frac{1}{3} \quad \text{and} \quad \lim_{x\to 1-} \frac{|x-1|}{x^2+x-2} = -\frac{1}{3}.$$

(d) $f(x) = \dfrac{x^2+x-2}{\sqrt{x+2}} = \sqrt{x+2}\,(x-1)$ is undefined if $x < -2$, so $\lim_{x\to -2-} f(x)$ does not exist. If $x > -2$ then $|f(x)| = \sqrt{x+2}\,|x-1| \le \sqrt{x+2}|x+2-3| \le \sqrt{x+2}\,(|x+2|+3)$. Therefore if $\delta \le \min(1, \epsilon^2/4)$ then $|f(x)| < \epsilon$ if $-2 < x < -2+\delta$. Hence $\lim_{x\to -2+} f(x) = 0$.

2.1.8. If $\lim_{x\to x_0-} h(x) = -\rho$ ($\rho > 0$), let $\epsilon = \rho/2$ in Definition 2.1.5**(a)**. Then there is a $\delta > 0$ such that $|h(x) - (-\rho)| < \rho/2$ if $x_0 - \delta < x < x_0$. This is impossible since $|h(x) - (-\rho)| = h(x) + \rho \ge \rho$ if $h(x) \ge 0$.

2.1.9. **(a)** Letting $\epsilon = 1$ in Definition 2.1.2, we see that there is a $\rho > 0$ such that $|f(x) - L| < 1$, so $|f(x)| < |L| + 1$, if $0 < |x - x_0| < \rho$. **(b)** For "$\lim_{x\to x_0-}$" the set is of the form $(x_0 - \rho, x_0)$. For "$\lim_{x\to x_0+}$" the set is of the form $(x_0, x_0 + \rho)$.

2.1.10. **(a)** Let P_n be the proposition that $\lim_{x\to x_0} [f(x)]^n = L^n$. P_1 is true by assumption. If $n \ge 1$ and P_n is true then $\lim_{x\to x_0} [f(x)]^{n+1} = \left(\lim_{x\to x_0} [f(x)]^n\right) \lim_{x\to x_0} f(x)$ (by Eqn. (2.1.12))$= L^n L$ (by P_n)$= L^{n+1}$, which proves P_n.

(b) In the identity $u^n - v^n = (u-v)\sum_{r=0}^{n-1} u^r v^{n-1-r}$, let $u = f(x)$ and $v = L$ to obtain (A) $[f(x)]^n - L^n = (f(x)-L)\sum_{r=0}^{n-1}[f(x)]^r L^{n-1-r}$. From Exercise 2.1.9, there is a $\rho > 0$ such that $|f(x)| < |L| + 1$ if $0 < |x - x_0| < \rho$; hence (A) implies that $|[f(x)]^n - L^n| < K|f(x) - L|$ if $0 < |x - x_0| < \rho$, where K is a constant. If $\epsilon > 0$ there is a $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ if $0 < |x - x_0| < \delta_1$. Therefore $|[f(x)]^n - L^n| < K\epsilon$ if $0 < |x - x_0| < \min(\rho, \delta_1)$.

2.1.11. Write $\left|\sqrt{f(x)} - \sqrt{L}\right| = |f(x) - L| \big/ \left(\sqrt{f(x)} + \sqrt{L}\right) \le |f(x) - L| \big/ \sqrt{L}$ and apply Definition 2.1.2.

2.1.12. Use Definitions 2.1.2 and 2.1.5.

2.1.14. $\lim_{x\to-\infty} f(x) = L$ if f is defined on an interval $(-\infty, b)$ and for each $\epsilon > 0$ there is a number α such that $|f(x) - L| < \epsilon$ if $x < \alpha$.

2.1.15. **(a)** $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$ because $\frac{1}{x^2+1} < \epsilon$ if $x > 1/\sqrt{\epsilon}$.

(b) $\lim_{x\to\infty} \frac{\sin x}{|x|^\alpha} = 0$ if $\alpha < 0$ because $\left|\frac{\sin x}{x^\alpha}\right| \le \frac{1}{x^\alpha} < \epsilon$ if $x > \frac{1}{\epsilon^{1/\alpha}}$.

(c) $\lim_{x\to\infty} \frac{\sin x}{x^\alpha}$ does not exist if $\alpha \le 0$ since, for example, $\frac{\sin x}{x^\alpha}$ assumes the values ± 1 on very interval $[a, \infty)$.

(d) $\lim_{x\to\infty} e^{-x} \sin x = 0$ because $|e^{-x}\sin x| = e^{-x} < \epsilon$ if $x > -\ln\epsilon$.

(e) $\lim_{x\to\infty} \tan x$ does not exist because $\tan x$ is not defined on $[a, \infty)$ for any a.

(f) $\lim_{x\to\infty} e^{-x^2} e^{2x} = 0$ because if $\epsilon < e$ then $e^{-x^2}e^{2x} = ee^{-(x-1)^2} < \epsilon$ if $x > 1 + \sqrt{\ln(e/\epsilon)}$.

2.1.16. For "$\lim_{x\to\infty}$" statements such as "there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < |x - x_0| < \delta$" would be changed to "there is an α such that $|f(x) - L| < \epsilon$ if $x > \alpha$,"; for "$\lim_{x\to-\infty}$" the last inequality would be replaced by "$x < \alpha$."

2.1.19. **(a)** $\lim_{x\to x_0-} f(x) = -\infty$ if f is defined on an interval (a, x_0) and for each real number M there is a $\delta > 0$ such that $f(x) < M$ if $x_0 - \delta < x < x_0$.

(b) $\lim_{x\to x_0+} f(x) = \infty$ if f is defined on an interval (x_0, b) and for every real number M there is a $\delta > 0$ such that $f(x) > M$ if $x_0 < x < x_0 + \delta$.

(c) $\lim_{x\to x_0+} f(x) = -\infty$ if f is defined on an interval (x_0, b) and for every real number M there is a $\delta > 0$ such that $f(x) < M$ if $x_0 < x < x_0 + \delta$.

2.1.21. **(a)** $\lim_{x \to x_0} f(x) = \infty$ if f is defined on a deleted neighborhood of x_0 and, for every real number M, there is a $\delta > 0$ such that $f(x) > M$ if $0 < |x - x_0| < \delta$.

(b) $\lim_{x \to x_0} f(x) = \infty$ if f is defined on a deleted neighborhood of x_0 and for, every real number M, there is a $\delta > 0$ such that $f(x) < M$ if $0 < |x - x_0| < \delta$.

2.1.23. **(a)** $\lim_{x \to \infty} f(x) = \infty$ if f is defined on an interval (a, ∞) and, for every real number M, there is an α such that $f(x) > M$ if $x > \alpha$.

(b) $\lim_{x \to -\infty} f(x) = -\infty$ if f is defined on an interval $(-\infty, b)$ and for, every real number M, there is an α such that $f(x) < M$ if $x < \alpha$.

2.1.25. Suppose that L is finite and $\epsilon > 0$. Since $\lim_{x \to \infty} f(x) = L$, there is an $\alpha \in (a, \infty)$ such that $|f(x) - L| < \epsilon$ if $x > \alpha$. Since $\lim_{x \to \infty} g(x) = \infty$, there is a $\gamma \in (c, \infty)$ such that $g(x) > \alpha$ if $x > \gamma$. Therefore $|f(g(x)) - L| < \epsilon$ if $x > \gamma$, so $\lim_{x \to \infty} f(g(x)) = L$.

Now suppose that $L = \infty$ and M is arbitrary. Since $\lim_{x \to \infty} f(x) = \infty$, there is an $\alpha \in (a, \infty)$ such that $f(x) > M$ if $x > \alpha$. Since $\lim_{x \to \infty} g(x) = \infty$, there is a $\gamma \in (c, \infty)$ such that $g(x) > \alpha$ if $x > \gamma$. Therefore $f(g(x)) > M$ if $x > \gamma$, so $\lim_{x \to \infty} f(g(x)) = \infty$.

The proof where $L = -\infty$ is similar to this.

2.1.26. **(a)** Suppose that $\lim_{x \to x_0} f(x) = L$, and choose $\delta > 0$ so $|f(x) - L| < \epsilon_0/2$ if $0 < |x - x_0| < \delta$. If x_1 and x_2 are in $(x_0 - \delta, x_0 + \delta)$ and distinct from x_0, then $|f(x_1) - f(x_2)| \le |f(x_1) - L| + |L - f(x_2)| < \epsilon_0$, a contradiction. Hence $\lim_{x \to x_0} f(x)$ does not exist.

(b) Replace "every deleted neighborhood of x_0" by "every interval (x_0, b)," "every interval (a, x_0)," "every interval (a, ∞)", and "every interval $(-\infty, b)$", respectively.

2.1.28. Let M be arbitrary. If $-\infty < L_1 < L_2 = \infty$, choose $\delta > 0$ so that $f(x) > L_1 - 1$ and $g(x) > M - L_2 + 1$ if $0 < |x - x_0| < \delta$; if $L_1 = L_2 = \infty$, choose $\delta > 0$ so that $f(x) > M/2$ and $g(x) > M/2$ if $0 < |x - x_0| < \delta$. Then $(f + g)(x) > M$ if $0 < |x - x_0| < \delta$, so $\lim_{x \to x_0} (f + g)(x) = \infty = L_1 + L_2$.

If $-\infty = L_1 < L_2 < \infty$, choose $\delta > 0$ so that $g(x) < L_2 + 1$ and $f(x) < M - L_2 - 1$ if $0 < |x - x_0| < \delta$; if $L_1 = L_2 = -\infty$ choose $\delta > 0$ so that $f(x) < M/2$ and $g(x) < M/2$ if $0 < |x - x_0| < \delta$. Then $(f + g)(x) < M$ if $0 < |x - x_0| < \delta$, so $\lim_{x \to x_0} (f + g)(x) = -\infty = L_1 + L_2$.

2.1.29. Suppose that $L_1 = \infty$, $0 < \rho < L_2 \le \infty$, and $M > 0$. Choose τ so $g(x) > \rho$ and $f(x) > M/\rho$ if $x > \tau$; then $(fg)(x) > M$ if $x > \tau$, so $\lim_{x \to \infty} (fg)(x) = \infty = L_1 \cdot L_2$. Similar arguments apply to the other cases.

2.1.30. **(a)** Let M be arbitrary. Suppose that $L_1 = \infty$. If $0 < L_2 < \infty$ choose $\delta > 0$ so $0 < g(x) < 3L/2$ and $f(x) > 2M/3L_2$ if $0 < |x - x_0| < \delta$. Then $(f/g)(x) > M$ if $0 < |x - x_0| < \delta$,so $\lim_{x \to x_0} (f/g)(x) = \infty = L_1/L_2$. Similar arguments apply to the other cases where $L_1 = \pm\infty$ and $0 < |L_2| < \infty$.

Now suppose that $|L_1| < \infty$ and $|L_2| = \infty$. If $\epsilon > 0$ choose $\delta > 0$ so $|f(x)| < |L_1| + 1$ and $|g(x)| > (|L_1| + 1)/\epsilon$ if $0 < |x - x_0| < \delta$. Then $|(f/g)(x)| < \epsilon$ if $0 < |x - x_0| < \delta$, so $\lim_{x\to x_0} (f/g)(x) = 0 = L_1/L_2$.

(b) $\left(\lim_{x\to\pi/2} \sin x\right) \Big/ \left(\lim_{x\to\pi/2} \cos x\right) = 1/0 = \infty$ is not indeterminate but $\lim_{x\to\pi/2+} \tan x = -\infty$ and $\lim_{x\to\pi/2-} \tan x = \infty$, so $\lim_{x\to\pi/2} \tan x$ does not exist in the extended reals.

2.1.32.

$$\lim_{x\to\infty} r(x) = \left(\lim_{x\to\infty} x^{n-m}\right)\left(\lim_{x\to\infty} \frac{a_0x^{-n} + a_1x^{-n+1} + \cdots + a_n}{b_0x^{-m} + b_1x^{-m+1} + \cdots + b_m}\right)$$

$$= \lim_{x\to\infty} \frac{a_n}{b_m}x^{n-m} = \begin{cases} 0 & \text{if } n < m, \\ \dfrac{a_n}{b_m} & \text{if } n = m, \\ \infty & \text{if } n > m \text{ and } a_n/b_m > 0, \\ -\infty & \text{if } n > m \text{ and } a_n/b_m < 0. \end{cases}$$

Similarly,

$$\lim_{x\to-\infty} r(x) = \lim_{x\to-\infty} \frac{a_n}{b_m}x^{n-m} = \begin{cases} 0 & \text{if } n < m, \\ \dfrac{a_n}{b_m} & \text{if } n = m, \\ (-1)^{n-m}\infty & \text{if } n > m \text{ and } a_n/b_m > 0, \\ (-1)^{n-m+1}\infty & \text{if } n > m \text{ and } a_n/b_m < 0. \end{cases}$$

2.1.33. $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$ for all x in (a,b). Proof: If $\lim_{x\to x_0} f(x) = L$ and $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < |x - x_0| < \delta$. Since x_0 is not a limit point of S, there is a δ_1 such that $0 < \delta_1 < \delta$ and $g(x) = f(x)$ if $|x - x_0| < \delta_1$. Therefore $|g(x) - L| < \epsilon$ if $0 < |x - x_0| < \delta_1$, so $\lim_{x\to x_0} g(x) = L$.

2.1.34. **(b)** We first prove that $f(a+) = \beta$. If $M < \beta$ there is an x_0 in (a,b) such that $f(x_0) > M$. Since f is nonincreasing, $f(x) > M$ if $a < x < x_0$. Therefore, if $\beta = \infty$ then $f(a+) = \infty$. If $\beta < \infty$ let $M = \beta - \epsilon$ where $\epsilon > 0$. Then $\beta - \epsilon < f(x) \le \beta + \epsilon$, so

$$|f(x) - \beta| < \epsilon \quad \text{if} \quad a < x < x_0. \tag{A}$$

If $a = -\infty$ this implies that $f(-\infty) = \beta$. If $a > -\infty$ let $\delta = x_0 - a$. Then (A) is equivalent to

$$|f(x) - \beta| < \epsilon \quad \text{if} \quad a < x < a + \delta,$$

which implies that $f(a+) = \beta$.

Now we prove that $f(b-) = \alpha$. If $M > \alpha$ there is an x_0 in (a,b) such that $f(x_0) < M$. Since f is nonincreasing, $f(x) < M$ if $x_0 < x < b$. Therefore, if $\alpha = -\infty$ then $f(b-) = -\infty$. If $\alpha > -\infty$ let $M = \alpha + \epsilon$ where $\epsilon > 0$. Then $\alpha \le f(x) < \alpha + \epsilon$, so

$$|f(x) - \alpha| < \epsilon \quad \text{if} \quad x_0 < x < b. \tag{B}$$

If $b = \infty$ this implies that $f(\infty) = \alpha$. If $b < \infty$ let $\delta = b - x_0$. Then (B) is equivalent to

$$|f(x) - \alpha| < \epsilon \quad \text{if} \quad b - \delta < x < b,$$

which implies that $f(b-) = \alpha$.

(c) Applying **(b)** to f on (a, x_0) and (x_0, b) separately shows that

$$f(x_0-) = \inf_{a<x_1<x_0} f(x_1) \quad \text{and} \quad f(x_0+) = \sup_{x_0<x_2<b} f(x_2).$$

However, if $x_1 < x_0 < x_2$ then $f(x_1) \ge f(x_0) \ge f(x_2)$; hence $f(x_0-) \ge f(x_0) \ge f(x_0+)$.

2.1.36. **(a)** $I_f(x; x_0) \le S_f(x; x_0)$, $a < x < x_0$, so applying Exercise 2.1.8 to $h(x) = S_f(x; x_0) - I_f(x; x_0)$ yields $\underline{\lim}_{x\to x_0-} f(x) \le \overline{\lim}_{x\to x_0-} f(x)$.

(b) $I_{-f}(x; x_0) = \inf_{x\le t<x_0} (-f(t)) = -\sup_{x\le t<x_0} f(t) = -S_f(x; x_0)$. Therefore

$$\underline{\lim}_{x\to x_0-} (-f)(x) = \lim_{x\to x_0-} I_{-f}(x; x_0) = -\lim_{x\to x_0-} S_f(x; x_0) = -\overline{\lim}_{x\to x_0-} f(x).$$

$S_{-f}(x; x_0) = \sup_{x\le t<x_0} (-f(t)) = -\inf_{x\le t<x_0} f(t) = -I_f(x; x_0)$. Therefore

$$\overline{\lim}_{x\to x_0-} (-f)(x) = \lim_{x\to x_0-} S_{-f}(x; x_0) = -\lim_{x\to x_0-e} I_f(x; x_0) = -\underline{\lim}_{x\to x_0} f(x).$$

(c) Let $\epsilon > 0$. Suppose that (A) $\underline{\lim}_{x\to x_0-} f(x) = \overline{\lim}_{x\to x_0-} f(x) = L$. Then there is an $\delta > 0$ such that

$$L - \epsilon \le I_f(x; x_0) \le S_f(x; x_0) \le L + \epsilon, \quad x_0 - \delta < x < x_0. \tag{B}$$

Since $I_f(x; x_0) \le f(x) \le S_f(x; x_0)$, this implies that

$$L - \epsilon \le f(x) \le L + \epsilon, \quad x_0 - \delta < x < x_0; \tag{C}$$

hence $\lim_{x\to x_0-} f(x) = L$. Conversely, if $\lim_{x\to x_0} f(x) = L$ then (C) holds for some $\delta > 0$. But (C) implies (B) and (B) implies that $L - \epsilon \le \underline{\lim}_{x\to x_0-} f(x) \le \overline{\lim}_{x\to x_0-} f(x) \le L + \epsilon$ (Exercise 2.1.8). This implies (A).

2.1.37. **(a)** $S_{f+g} = \sup_{x\le t<x_0-} (f(t)+g(t)) \le \sup_{x\le t<x_0-} f(t) + \sup_{x\le t<x_0-} g(t) = S_f(x; x_0) + S_g(x; x_0)$. Therefore $\overline{\lim}_{x\to x_0-} (f + g)(x) = \lim_{x\to x_0-} S_{f+g}(x; x_0) \le \lim_{x\to x_0-} S_f(x; x_0) + \lim_{x\to x_0-} S_g(x; x_0) = \overline{\lim}_{x\to x_0-} f(x) + \overline{\lim}_{x\to x_0-} g(x)$

(b) Applying **(a)** to $-f$ and $-g$ yields

$$\overline{\lim}_{x\to x_0-} (-(f + g))(x) \le \overline{\lim}_{x\to x_0-} (-f)(x) + \overline{\lim}_{x\to x_0-} (-g)(x).$$

Now Exercise 2.1.36(b) implies that

$$-\varliminf_{x\to x_0-}(f+g)(x) \le -\varliminf_{x\to x_0-} f(x) - \varliminf_{x\to x_0-} g(x),$$

so

$$\varliminf_{x\to x_0-}(f+g)(x) \ge \varliminf_{x\to x_0-} f(x) + \varliminf_{x\to x_0-} g(x).$$

(c) Write $f-g = f+(-g)$, and use **(a)** and **(b)** and Exercise 2.1.36(**b**) to show that

$$\varlimsup_{x\to x_0-}(f-g)(x) \le \varlimsup_{x\to x_0-} f(x) - \varliminf_{x\to x_0-} g(x)$$

and

$$\varliminf_{x\to x_0-}(f-g)(x) \ge \varliminf_{x\to x_0-} f(x) - \varlimsup_{x\to x_0-} g(x).$$

2.1.38. Necessity: If $\lim_{x\to x_0-} f(x) = L$ and $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon/2$ if $x_0 - \delta < x < x_0$; hence $|f(x_1) - f(x_2)| < |f(x_1) - L| + |f(x_2) - L| < \epsilon$ if $x_0 - \delta < x_1, x_2 < x_0$. Sufficiency: First let $\epsilon = 1$ and note that there is a $\delta_0 > 0$ such that $|f(x_1) - f(x_2)| < 1$ if $x_0 - \delta_0 < x_1, x_2 < x_0$. Choose a in $(x_0 - \delta_0, x_0)$. Then $|f(x)| \le 1 + |f(a)|$ if $a < x < x_0$; that is, f is bounded on (a, x_0). Now suppose that $\epsilon > 0$, and choose $\delta < \delta_0$ so that $|f(x_1) - f(x_2)| < \epsilon$ if $x_0 - \delta,\ x_1, x_2 < x_0$. Then $0 \le S_f(x; x_0) - I_f(x; x_0) < \epsilon$ if $x_0 - \delta < x < x_0$. Letting $x \to x_0-$ yields $0 \le \varlimsup_{x\to x_0-} f(x) - \varliminf_{x\to x_0-} f(x) < \epsilon$, which implies that $\varlimsup_{x\to x_0-} f(x) = \varliminf_{x\to x_0-} f(x)$. Hence $\lim_{x\to x_0} f(x)$ exists (finite) by Exercise 2.1.36(**c**).

2.1.39. For $x_0 < x < b$, define $S_f(x; x_0) = \sup_{x_0 \le t < x} f(t)$ and $I_f(x; x_0) = \inf_{x \le t < x_0} f(t)$. Then $\varlimsup_{x\to x_0+} f(x) = \lim_{x\to x_0+} S_f(x; x_0)$ and $\varliminf_{x\to x_0-} f(x) = \lim_{x\to x_0-} I_f(x; x_0)$. The existence proofs are similar to those in Theroems 2.1.11 and 2.1.12.

2.1.40. Similar to the solution of Exercise 2.1.36(**c**).

2.1.41. Use Theorem 2.1.6 (in the extended reals), and Exercise 36(**c**) and Exercise 2.1.40.

2.2 CONTINUITY

2.2.1. Just invoke Definitions 2.1.2 and 2.1.5 with $L = f(x_0)$.

2.2.2. Apply Theorem 2.1.6) $L = \lim_{x\to x_0-} f(x)$, $L = \lim_{x\to x_0+} f(x)$, and $L = f(x_0)$.

2.2.6. Let x_0 be an arbitrary real number. Since every interval contains both rational and irrational numbers Theorems 1.1.6 and (1.1.7)), every interval containing x_0 contains a point x such that $|f(x) - f(x_0)| = 2$. Therefore $\lim_{x\to x_0}$ does not exist, so f is not continuous at x_0.

2.2.7. If $x_0 = p/q$ where p and q are integers with no common factor, then $f(x_0) = 1/q$, while every neighborhood of x_0 contains a number x (irrational) such that $|f(x) - f(x_0)| = 1/q$. Therefore f is discontinuous at every rational. If x_0 is irrational, then $f(x_0) = 0$.

Given $\epsilon > 0$, choose $\delta > 0$ so small that the interval $(x_0 - \delta, x_0 + \delta)$ contains no rational p/q with $q < 1/\epsilon$. If x is in this interval, then $|f(x) - f(x_0)| = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = p/q. \end{cases}$.
In either case $|f(x) - f(x_0)| < \epsilon$. Hence, f is continuous at every irrational.

2.2.8. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the distinct values of f. Suppose $f(x_0) = \lambda_i$ and

$$0 < \epsilon < \min\left\{|\lambda_i - \lambda_j| \,\middle|\, i \le j \le n,\ j \ne i\right\}.$$

If f is continuous at x_0, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$. Therefore $f(x) = f(x_0)$ if $|x - x_0| < \delta$. The converse is obvious.

2.2.9. Suppose ψ_T is continuous at x_0. Then Exercise 2.2.8 implies that ψ_T is constant on some interval $I = (x_0 - \delta, x_0 + \delta)$. Therefore $I \subset T$ (so $x_0 \in T^0$) if $x_0 \in T$, and $I \subset T^c$ (so $x_0 \in (T^c)^0$) if $x_0 \in T^c$.

Conversely, suppose that $x_0 \in T^0$. Then $I = (x_0 - \delta, x_0 + \delta) \subset T^0$ for some $\delta > 0$, so $\psi_T(x) = 1$ for all $x \in I$, which implies that f is continuous at x_0. Now suppose $x_0 \in (T^c)^0$. Then $I = (x_0 - \delta, x_0 + \delta) \subset (T^c)^0$ for some $\delta > 0$, so $\psi_T(x) = 0$ for all $x \in I$, which again implies that f is continuous at x_0. Hence, f is continuous at every x_0 in $T^0 \cup (T^c)^0$.

2.2.10. Let $h = f - g$ and suppose $x_0 \in (a, b)$. We must show that $h(x_0) = 0$. Let $\epsilon > 0$. Since h is continuous on (a, b), there is a $\delta > 0$ such that (A) $|h(x) - h(x_0)| < \epsilon$ if $|x - x_0| < \delta$. By assumption, $h(x) = 0$ for some x in $(x_0 - \delta, x_0 + \delta)$, so (A) implies that $|h(x_0)| < \epsilon$. Since this holds for every $\epsilon > 0$, $h(x_0) = 0$.

2.2.11. Suppose $\epsilon > 0$. From **(a)** there is a $\rho > 0$ such that (A) $|g(u)| < \epsilon$ if $0 < |1 - u| < \rho$. If $x, x_0 > 0$ and $|x - x_0| < \delta =_{\text{df}} \rho x_0$ then $|1 - x/x_0| < \rho$, so

$$\begin{aligned} |g(x) - g(x_0)| &= |g\,[x_0(x/x_0)] - g(x_0)| \\ &= |g(x/x_0) + g(x_0) - g(x_0)| \qquad \text{(from (b))} \\ &= |g(x/x_0)| < \epsilon \qquad \text{(from (A))}. \end{aligned}$$

Hence, g is continuous at x_0.

2.2.12. Suppose $\epsilon > 0$. From **(a)** there is a $\delta > 0$ such that (A) $|f(u) - 1| < \epsilon$ if $0 < |u| < \delta$. If $x, x_0 > 0$ then **(b)** with $x_1 = x - x_0$ and $x_2 = x_0$ implies that $f(x) = f(x_0)f(x - x_0)$. Therefore $|f(x) - f(x_0)| \le |f(x_0)|\,|f(x - x_0) - 1|$. Now (A) with $u = x - x_0$ implies that $|f(x) - f(x_0)| < |f(x_0)|\epsilon$ if $|x - x_0| < \delta$. Hence, f is continuous at x_0.

2.2.13. **(a)** Write $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$; then use Exercise 2.2.12 and Theorem 2.2.5.

(b) $\tanh x$ is continuous for all x, $\coth x$ for all $x \ne 0$.

2.2.14. Suppose $\epsilon > 0$. From **(a)** there is a $\delta > 0$ such that (A) $|c(u) - 1| < \epsilon$ if $0 < |u| < \delta$. If $x, x_0 > 0$ then **(b)** and **(c)** imply that $(s(x) - s(x_0))^2 + (c(x) - c(x_0))^2 = 2(1 - c(x - x_0))$; hence, $|s(x) - s(x_0)| \le \sqrt{2(1 - c(x - x_0))}$ and $|c(x) - c(x_0)| \le \sqrt{2(1 - c(x - x_0))}$. Therefore (A) implies that $|s(x) - s(x_0)| < \sqrt{2}\epsilon$ and $|c(x) - c(x_0)| < \sqrt{2}\epsilon$ if $|x - x_0| < \delta$. Hence, c and s are continuous at x_0.

2.2.15. **(a)** If $\epsilon = (f(x_0) - \mu)/2$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$. Therefore $f(x) \ge f(x_0) - \epsilon = (f(x_0) + \mu)/2 > \mu$ if $|x - x_0| < \delta$ and $x \in D_f$.

(b) Replace "$>$" by "$<$" in **(a)**.

(c) If $f(x_0) > \mu$, **(a)** implies that $f(x) > \mu$ in some neighborhood of x_0. This contradicts the assumption that $f(x) \le \mu$ for all x.

2.2.16. **(a)** Suppose $\epsilon > 0$. Since f is continuous at x_0, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$. Since $\big||f(x)| - |f(x_0)|\big| \le |f(x) - f(x_0)|$, it follows that $\big||f(x)| - |f(x_0)|\big| < \epsilon$ if $|x - x_0| < \delta$. Therefore $|f|$ is continuous at x_0.

(b) No; in Exercise 2.2.6, $|f|$ is continuous and f is discontinuous for all x.

2.2.17. Theorem 2.1.9 implies **(a)** and **(b)** of Definition 2.2.4, so any discontinuities of f must be jump discontinuities. The only remaining requirement of Definition 2.2.4 is that there be only finitely many of these.

2.2.18. See the proof of Theorem 2.1.4.

2.2.19. **(a)** The proposition is true for $n = 2$, by Theorem 2.2.5. Suppose it is true for some $n \ge 2$ and let $f_1, f_2, \dots, f_{n+1}$ be continuous on S. Since $f_1 + f_2 + \cdots + f_{n+1} = (f_1 + f_2 + \cdots + f_n) + f_{n+1}$, Theorem 2.2.5 with $f = f_1 + f_2 + \cdots + f_n$ and $g = f_{n+1}$ implies that $f_1 + f_2 + \cdots + f_{n+1}$ is continuous on S. Since $f_1 f_2 \cdots f_{n+1} = (f_1 f_2 \cdots f_n) f_{n+1}$, Theorem 2.2.5 with $f = f_1 f_2 \cdots f_n$ and $g = f_{n+1}$ implies that $f_1 f_2 \cdots f_{n+1}$ is continuous on S. This completes the induction.

(b) Since ax^k is continuous everywhere if a is a constant and k is a nonnegative integer, **(a)** implies that $a_0 + a_1 x + \cdots + a_n x^n$ and $b_0 + b_1 x + \cdots + b_m x^m$ are both continuous everywhere. Therefore Theorem 2.2.5 implies that $r(x) = (a_0 + a_1 x + \cdots + a_n x^n)/(b_0 + b_1 x + \cdots + b_m x^m)$ is continuous wherever $b_0 + b_1 x + \cdots + b_m x^m \ne 0$.

2.2.20. **(a)** Suppose $\epsilon > 0$. There are two cases to consider: (I) $f_1(x_0) = f_2(x_0)$; and (II) $f_1(x_0) \ne f_2(x_0)$.

CASE I. If $f_1(x_0) = f_2(x_0) = K$, choose $\delta > 0$ so that $|f_i(x) - K| < \epsilon$ $(i = 1, 2)$ if $|x - x_0| < \delta$. Since for any x either $|F(x) - F(x_0)| = |f_1(x) - K|$ or $|F(x) - F(x_0)| = |f_2(x) - K|$, it follows that $|F(x) - F(x_0)| < \epsilon$ if $|x - x_0| < \delta$.

CASE II. If $f_1(x_0) \ne f_2(x_0)$, assume without loss of generality that $f_2(x_0) > f_1(x_0)$. Since $f_2 - f_1$ is continuous at x_0, Exercise 2.2.15**(a)** implies that there is a $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$ then $f_2(x) > f_1(x)$, and therefore $F_2(x) = f_2(x)$. Now choose δ_2 so that $0 < \delta_2 < \delta_1$ and $|f_2(x) - f_2(x_0)| < \epsilon$ if $|x - x_0| < \delta_2$. Now $|F(x) - F(x_0)| = |f_2(x) - f_2(x_0)| < \epsilon$ if $|x - x_0| < \delta_2$.

(b) Use induction. From **(a)**, P_2 is true. Suppose that $n \ge 2$ and P_n is true. Let $F(x) = \max(f_1(x), \dots, f_{n+1}(x))$, where $f_1, \dots, f_{n+1}$ are all continuous at x_0. Then $F(x) = \max(g(x), f_{n+1}(x))$, with $g(x) = \max(f_1(x), \dots, f_{n-1}(x))$, which is continuous at x_0 by the induction assumption. Applying P_2 to f_n and g shows that F is continuous at x_0, which proves P_{n+1}.

2.2.22. **(a)** Since f is continuous at y_0, there is $\epsilon_1 > 0$ such that (A) $|f(t) - f(y_0)| < \epsilon$ if $|t - y_0| < \epsilon_1$. Since $y_0 = \lim_{x \to x_0} g(x)$, there is a $\delta > 0$ such that (B) $0 < |g(x) - y_0| < \epsilon_1$ if $0 < |x - x_0| < \delta$. Now (A) and (B) imply that $|f(g(x)) - f(g(x_0))| < \epsilon$ if $0 < |x - x_0| < \delta$.

Therefore $\lim_{x \to x_0} f(g(x)) = y_0$.

2.2.24. Suppose there is no x_2 in $[a,b]$ such that $f(x_2) = \beta$. Then $f(x) < \beta$ for all $x \in [a,b]$. We will show that this leads to a contradiction. Suppose $t \in [a,b]$. Then $f(t) < \beta$, so $f(t) < (f(t) + \beta)/2 < \beta$. Since f is continuous at t, there is an open interval I_t about t such that (A) $f(x) < (f(t) + \beta)/2)$ if $x \in I_t \cap [a,b]$ (Exercise 2.2.15). The collection $H = \left\{ I_t \,\middle|\, a \le t \le b \right\}$ is an open covering of $[a,b]$. Since $[a,b]$ is compact, the Heine-Borel theorem implies that there are finitely many points $t_1, t_2, \dots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \dots, I_{t_n}$ cover $[a,b]$. Define $\beta_1 = \max \left\{ (f(t_i) + \beta)/2 \,\middle|\, 1 \le i \le n \right\}$. Then, since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$, (A) implies that $f(x) < \beta_1$ $(a \le t \le b)$. But $\beta_1 < \beta$, so this contradicts the definition of β. Therefore $f(x_2) = \beta$ for some x_2 in $[a,b]$.

2.2.25. Let $\alpha = \inf S$ and $\beta = \sup S$. Then $\alpha < \beta$, since f is nonconstant. We first show that $(\alpha, \beta) \subset S$. If $\alpha < y < \beta$, there are points x_1, x_2 in I such that $\alpha < f(x_1) < y < f(x_2) < \beta$, by definition of α and β. Applying Theorem 2.2.10 to $[x_1, x_2]$ shows that $f(c) = y$ for some c between x_1 and x_2. Therefore $(\alpha, \beta) \subset S$. Since $y \notin S$ if $y < \alpha$ or $y > \beta$, S is one of the intervals (α, β), $[\alpha, \beta)$, $(\alpha, \beta]$, or $[\alpha, \beta]$. (If $\alpha = -\infty$ or $\beta = \infty$, this statement must be modified in the obvious way.) If I is a finite closed interval, then f attains the values α and β on I (Theorem 2.2.9), so $S = [\alpha, \beta]$.

2.2.26. Let x_0 be arbitrary and suppose $\epsilon > 0$. Since f is increasing, $f(g(x_0) - \epsilon) < f(g(x_0)) < f(g(x_0) + \epsilon)$. Let

$$\epsilon_1 = \min\{ f(g(x_0)) - f(g(x_0) - \epsilon), \quad f(g(x_0) + \epsilon) - f(g(x_0)))\} .$$

Then $\epsilon_1 > 0$. Since $f \circ g$ is continuous at x_0, there is a $\delta > 0$ such that $|f(g(x)) - f(g(x_0))| < \epsilon_1$ if $|x - x_0| < \delta$. This implies that $f(g(x_0) - \epsilon) < f(g(x)) < f(g(x_0) + \epsilon)$ if $|x - x_0| < \delta$. Since f is increasing, this means that $g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$, or, equivalently, $|g(x) - g(x_0)| < \epsilon$ if $|x - x_0| < \delta$. Hence, g is continuous at x_0.

2.2.27. Since f is continuous on $[a,x]$ $(a \le x < b)$, Theorem 2.2.9 implies that F is well defined on $[a,b)$. Suppose $\epsilon > 0$. Since f is continuous from the right at a, there is a $\delta > 0$ such that $|f(t) - f(a)| < \epsilon$ if $a \le x < a + \delta$. Since $F(a) = f(a)$, this implies that $|f(t) - F(a)| < \epsilon$, so $f(t) < F(a) + \epsilon$, if $a \le t \le x < a + \delta$. Therefore $F(a) \le F(x) < F(a) + \epsilon$ if $a \le x < a + \delta$, so F is continuous from the right at a.

Now suppose $a < x_0 < b$. If $F(x_0) > f(x_0)$ then $F(x_0) > f(x)$ for x in some neighborhood N of x_0, so $F(x) = F(x_0)$ in N; hence, F is continuous at x_0. If $F(x_0) = f(x_0)$ and $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - F(x_0)| < \epsilon$ if $|x - x_0| < \delta$. Then (A) $F(x_0) - \epsilon < f(x) < F(x_0) + \epsilon$ if $|x - x_0| < \delta$. Therefore $F(x_0) \le f(t) < F(x_0) + \epsilon$ if $x_0 \le t \le x < x_0 + \delta$, so $F(x_0) \le F(x) < F(x_0) + \epsilon$ if $x_0 \le x \le x_0 + \delta$. This implies that (B) $F(x_0+) = F(x_0)$. (A) also implies that $F(x_0) - \epsilon < f(t) \le F(x_0)$ if $x_0 - \delta < t \le x \le x_0$. Therefore $F(x_0) - \epsilon < F(x) \le F(x_0)$ if $x_0 - \delta < x \le x_0$. This implies that (C) $F(x_0-) = F(x_0)$. (B) and (C) imply that F is continuous at x_0.

2.2.28. For convenience we state the following definition of uniform continuity: *Let K be any given positive constant. Then f is uniformly continuous on a set S if for each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x')| < K\epsilon$ if $x, x' \in S$ and $|x - x'| < \delta$.* This is equivalent to Definition 2.2.11.

For **(a)**, **(b)** and **(c)**, suppose that $\epsilon > 0$ and $|f(x) - f(x')| < \epsilon$ and $|g(x) - g(x')| < \epsilon$ if

$|x - x'| < \delta$ and $x, x' \in S$.

(a) $|(f \pm g)(x) - (f \pm g)(x')| \le |f(x) - f(x')| + |g(x) - g(x')|) < 2\epsilon$ if $|x - x'| < \delta$ and $x, x' \in S$.

(b) By Theorem 2.2.8, there is an M such that $|f(x)| \le M$ and $g(x)| \le M$ for all x in S. Therefore

$$\begin{aligned} |(fg)(x) - (fg)(x')| &\le |f(x)g(x) - f(x')g(x)| + |f(x')g(x) - f(x')g(x')| \\ &\le M(|f(x) - f(x')| + |g(x) - g(x')| < 2M\epsilon \end{aligned}$$

if $|x - x'| < \delta$ and $x, x' \in S$.

(c) Since $|g|$ is continuous and nonzero on S, Theorem 2.2.9 implies that there is an $m > 0$ such that $|g(x)| \ge m$ for all x in S. Now,

$$\begin{aligned} |(f/g)(x) - (f/g)(x')| &= \left| \frac{f(x)g(x') - f(x')g(x)}{g(x)g(x')} \right| \\ &= \frac{|[f(x)g(x') - f(x')g(x')] + [f(x')g(x') - f(x')g(x)]|}{|g(x)g(x')|} \\ &\le \frac{|g(x')|\,|f(x) - f(x')| + |f(x')|\,|g(x') - q(x)|}{|g(x)g(x')|} \\ &\le 2M\epsilon/m^2 \qquad \text{(with } M \text{ as in } \textbf{(b)}) \end{aligned}$$

if $|x - x'| < \delta$ and $x, x' \in S$.

(d) For **(b)** take $f(x) = g(x) = x$; then f and g are uniformly continuous on $(-\infty, \infty)$ (Example 2.2.13), but $(fg)(x) = x^2$ (Example 2.2.15), which is not. For **(c)** take $f(x) = 1$, $g(x) = x$, and $I = (0, 1]$.

(e) For fg, require that f and g be bounded on S. For f/g, require this and also that $|g(x)| \ge \rho > 0$ for all x in S.

2.2.29. Suppose $\epsilon > 0$. Since f is uniformly continuous on S, there is an $\delta_1 > 0$ such that

$$|f(y) - f(y')| < \epsilon \quad \text{if} \quad |y - y'| < \delta_1 \quad \text{and} \quad y, y' \in S. \tag{A}$$

Since g is uniformly continuous on T, there is a $\delta > 0$ such that (B) $|g(x) - g(x')| < \delta_1$ if $|x - x'| < \delta$ and $y, y' \in T$. Now (A) and (B) imply that $|f(g(x)) - f(g(x'))| < \epsilon$ if $|x - x'| < \delta$ and $y, y' \in T$. Therefore $f \circ g$ is uniformly continuous on T.

2.2.30. **(a)** Let $I_j = [a_j, b_j]$, where $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, and $\delta_0 = \min\left\{a_{j+1} - b_j \,\middle|\, 1 \le j \le n-1\right\}$. If $\epsilon > 0$ there is a $\delta_j > 0$ such that $|f(x) - f(x')| < \epsilon$ if x and $x' \in I_j$ and $|x - x'| < \delta_j$. Let $\delta = \min\{\delta_0, \delta_1, \dots, \delta_n\}$. If x and x' are in $\cup_{j=1}^n I_j$ and $|x - x'| < \delta$, then x and x' are in the same I_j and $|f(x) - f(x')| < \epsilon$.

(b) No. If $f(x) = 1$ for $0 \le x \le 1$ and $f(x) = 0$ for $1 < x \le 2$, then f is uniformly continuous on $[0, 1]$ and $(1, 2]$ but not on $[0, 2]$.

2.2.31. **(b)** Consider $f(x) = x$ on $(-\infty, \infty)$.

2.2.32. Suppose $\lim_{x\to\infty} f(x) = L$ and $\epsilon > 0$. Choose R so that $|f(x) - L| < \epsilon/2$ if $x \ge R$. Then $|f(x_1) - f(x_2)| < \epsilon$ if $x_1, x_2 \ge R$. Since f is uniformly continuous on

$[a, 2R]$, there is a $\delta > 0$ such that $0 < \delta < R$ and $|f(x_1) - f(x_2)| < \epsilon$ if $|x_1 - x_2| < \delta$ and $a \le x_1, x_2 \le 2R$. If $a \le x_1, x_2$ and $|x_1 - x_2| < \delta$ then either $a \le x_1, x_2 \le 2R$ or $x_1, x_2 \ge R$. In either case $|f(x_1) - f(x_2)| < \epsilon$.

2.2.33. **(a)** Since $f(x) = [f(x/2)]^2$ from **(i)**, $f(x) \ge 0$ for all x. If $f(x_0) = 0$, then $f(x) = f(x_0)f(x - x_0) = 0$ for all x, which contradicts **(i)**. Threfore $f(x) > 0$ for all x.

(b) From (ii) and induction, (A) $f(mt) = (f(t))^m$ if m is a positive integer. Since $f(0) = 1$ because of **(i)** and continuity (Exercise 2.2.12), (A) holds for $m = 0$. If m is a negative integer, then $1 = f(0) = f(mt + |m|t) = f(mt)f(|m|t) = f(mt)(f(t))^{|m|}$; hence, $f(mt) = (f(t))^{-|m|} = (f(t))^m$. Hence, (A) holds for all integers m. If $r = p/q$ then $f(qrx) = (f(rx))^q$ ($t = rx$ and $m = q$ in (A)), while $f(qrx) = f(px) = (f(x))^p$ ($t = x, m = p$ in (A)). Therefore, $(f(x))^p = f(rx))^q$, so $f(rx) = (f(x))^{p/q} = (f(x))^r$.

(c) If $f(1) = 1$, **(a)** implies $f(r) = 1$ for every rational r. Since f is continuous on $(-\infty, \infty)$ (Exercise 2.2.12) and the rationals are dense in $(-\infty, \infty)$, $f \equiv 1$ on $(-\infty, \infty)$ (Exercise 2.2.10).

(d) Let $x_1 < r_1 < r_2 < x_2$, with r_1 and r_2 rational, and define $\epsilon_0 = \rho^{r_2} - \rho^{r_1} = f(r_2) - f(r_1) > 0$ (by (a)). By continuity, there are rationals r_1' and r_2' such that $x_1 < r_1' < r_1$ and $f(x_1) < f(r_1') + \epsilon_0/2$ and $r_2 < r_2' < x_2$ and $f(x_2) > f(r_2') - \epsilon_0/2$. Therefore $f(x_2) - f(x_1) > f(r_2') - f(r_1') - \epsilon_0 = (f(r_2') - f(r_2)) + (f(r_1) - f(r_1') > 0$ so f is increasing. If $A > 0$ then $f(r_0) = \rho^{r_0} > A$ for some rational r_0. Since f is increasing, $f(x) > A$ if $x \ge \rho_0$. Hence, $\lim_{x\to\infty} f(x) = \infty$. If $\epsilon > 0$ there is a negative rational r_1 such that $f(r_1) = \rho^{r_1} < \epsilon$. If $r < x < r_1$ (r rational) then $0 < \rho^r = f(x) < f(r_1) < \epsilon$. Hence, $0 < f(x) < \epsilon$ if $x < r_1$, so $\lim_{x\to\infty} f(x) = 0$.

2.2.34. Theorem 2.1.9**(b)** implies that the set $\widetilde{R}_f = \left\{f(x) \,\middle|\, x \in (a, b)\right\}$ is a subset of the open interval $(f(b-), f(a+))$. Therefore

$$R_f = \{f(b)\} \cup \widetilde{R}_f \cup \{f(a)\} \subset \{f(b)\} \cup (f(b-), f(a+)) \cup \{f(a)\}. \qquad \text{(A)}$$

Now suppose f is continuous on $[a, b]$. Then $f(a) = f(a+)$, $f(b-) = f(b)$, so (A) implies that $R_f \subset [f(b), f(a)]$. If $f(b) < \mu < f(a)$, then Theorem 2.2.10 implies that $\mu = f(x)$ for some x in (a, b). Hence, $R_f = [f(b), f(a)]$.

For the converse, suppose $R_f = [f(b), f(a)]$. Since $f(a) \ge f(a+)$ and $f(b-) \ge f(b)$, (A) implies that $f(a) = f(a+)$ and $f(b-) = f(b)$. We know from Theorem 2.1.9**(c)** that if f is nonincreasing and $a < x_0 < b$, then $f(x_0-) \ge f(x_0) \ge f(x_0+)$. If either of these inequalities is strict, then R_f cannot be an interval. Since this contradicts our assumption, $f(x_0-) = f(x_0) = f(x_0+)$. Therefore f is continuous at x_0 (Exercise 2.2.2). We can now conclude that f is continuous on $[a, b]$.

2.3 DIFFERENTIABLE FUNCTIONS OF ONE VARIABLE

2.3.1. Since $\dfrac{f(x)-f(x_0)-m(x-x_0)}{x-x_0} = \dfrac{f(x)-f(x_0)}{x-x_0} - m$, the conclusion follows from Definition 2.3.1.

2.3.2. $\lim_{x\to x_0} h(x)$ exists if and only if $f'(x_0)$ exists.

2.3.3. From Lemma 2.3.2, (A) $f(x)-f(x_0) = [f'(x_0)+E(x)](x-x_0)$ where $\lim_{x\to x_0} E(x) = E(x_0) = 0$. Choose $\delta > 0$ so that $|E(x)| < f'(x_0)$ if $0 < |x-x_0| < \delta$. Then $f'(x_0) + E(x) > 0$ if $0 < |x-x_0| < \delta$, so (A) implies that $f(x) > f(x_0)$ if $x_0 < x < x_0+\delta$ and $f(x) < f(x_0)$ if $x_0-\delta < x < x_0$.

2.3.4. **(a)** If $a < x < c$ and $|h| < \min(x-a, c-x)$, then $f(x+h)-f(x) = p(x+h) - p(x)$, so $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{p(x+h)-p(x)}{h} = p'(x)$.
If $c < x < b$ and $|h| < \min(x-c, b-x)$, then $f(x+h)-f(x) = q(x+h)-q(x)$, so $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{q(x+h)-q(x)}{h} = q'(x)$.
(b) If $f'(c)$ exists, then f must be continuous at c This is true if and only if $p(c) = q(c)$. If this condition holds then $f'(c)$ exists if and only if $p'_-(c) = q'_+(c)$.

2.3.5. Note that if k is a nonnegative integer, then $x^k|x| = \begin{cases} x^{k+1} & \text{if } x > 0, \\ -x^{k+1} & \text{if } x < 0. \end{cases}$ Apply Exercise 2.3.4.

2.3.6. Since $f(x) = f(x)f(0)$, either $f \equiv 0$ or $f(0) = 1$. The conclusion holds if $f \equiv 0$. If $f(0) = 1$ and x_0 is arbitrary, then $\lim_{h\to 0}(f(x_0+h) - f(x_0))/h = f(x_0)\lim_{h\to 0}(f(h) - 1)/h = f(x_0)f'(0)$.

2.3.7. **(a)** Set $y = 0$ to obtain

$$\begin{aligned} [c(0)-1]c(x) - s(0)s(x) &= 0 \\ s(0)c(x) + [c(0)-1]s(x) &= 0. \end{aligned}$$

Since $c^2 + s^2 \not\equiv 0$ (because $a^2+b^2 \neq 0$), the determinant of this system, $[c(0)-1]^2 + [s(0)]^2$, equals zero. Hence $c(0) = 1$, $s(0) = 0$, so $\dfrac{\lim_{h\to 0} c(h) - 1h}{=} \lim_{h\to 0}\dfrac{c(h)-c(0)}{h} = a$ and $\lim_{h\to 0}\dfrac{s(h)}{h} = \lim_{h\to 0}\dfrac{s(h)-s(0)}{h} = b$. Therefore,

$$\lim_{h\to 0}\frac{c(x+h)-c(x)}{h} = c(x)\lim_{h\to 0}\left(\frac{c(h)-1}{h}\right) - s(x)\lim_{h\to 0}\frac{s(h)}{h}.$$

so (A) $c'(x) = ac(x) - bs(x)$. Similarly,

$$\lim_{h\to 0}\frac{s(x+h)-s(x)}{h} = s(x)\lim_{h\to 0}\left(\frac{c(h)-1}{h}\right) - s(x)\lim_{h\to 0}\frac{s(h)}{h},$$

so (B) $s'(x) = bc(x) + as(x)$.

(b) The system (A) and (B) in **(a)** can be solved subject to the initial conditions $c(0) = 1$ and $s(0) = 0$ to obtain $c(x) = e^{ax}\cos bx$, $s(x) = e^{ax}\sin bx$.

2.3.8. **(a)** Write

$$\frac{f(x)}{g(x)} = \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \Big/ \left(\frac{g(x) - g(x_0)}{x - x_0}\right),$$

and use Theorem 2.1.4.

(b) If $f'_+(x_0)$ and $g'_+(x_0)$ exist, $f(x_0) = g(x_0) = 0$, and $g'_+(x_0) \neq 0$, then $\lim_{x\to x_0+} (f(x)/g(x)) = f'_+(x_0)/g'_+(x_0)$. A similar result holds with "–" replacing "+".

2.3.9.

$$\begin{aligned}\lim_{x\to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) + g'(x_0).\end{aligned}$$

2.3.10. See the solution of Exercise 2.3.9.

2.3.11.

$$\begin{aligned}\lim_{x\to x_0} \frac{(f/g)(x) - (f/g)(x_0)}{(x - x_0)} &= \lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{(x - x_0)g(x)g(x_0)} \\ &= \lim_{x\to x_0} \frac{[f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)]}{(x - x_0)g(x)g(x_0)} \\ &= \lim_{x\to x_0} \frac{1}{g(x)} \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &\quad + \frac{f(x_0)}{g(x_0)} \lim_{x\to x_0} \frac{1}{g(x)} \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.\end{aligned}$$

2.3.12. P_1 is true, by Theorem 2.3.4. Suppose that $n \geq 1$ and P_n is true. If $f^{(n+1)}(x_0)$ and $g^{(n+1)}(x_0)$ exist, then $f^{(n)}$ and $g^{(n)}$ exist on some neighborhood N of x_0, P_n implies that

$$(fg)^{(n)}(x) = \sum_{m=0}^{n} \binom{n}{m} f^{(k)}(x)g^{(n-m)}(x) \quad \text{if} \quad x \in N. \tag{A}$$

Theorem 2.3.4 implies that

$$\frac{d}{dx}\left(f^{(k)}(x)g^{(n-k)}(x)\right) = f^{(k+1)}(x)g^{(n-k)}(x) + f^{(k)}(x)g^{(n-k+1)}(x), \quad x \in N.$$

Substituting this into (A) and rearranging terms yields

$$\begin{aligned}(fg)^{(n+1)}(x) &= \sum_{m=0}^{n}\binom{n}{m}\left[f^{(m+1)}(x)g^{(n-m)}(x) + f^{(m)}(x)g^{(n-m+1)}(x)\right]\\ &= \sum_{m=1}^{n+1}\binom{n}{m-1}f^{(m)}(x)g^{(n-m+1)}(x) + \sum_{m=0}^{n}\binom{n}{m}f^{(m)}(x)g^{(n-m)+1}(x)\\ &= f(x)g^{(n+1)}(x) + \sum_{m=1}^{n}\left[\binom{n}{m-1}+\binom{n}{k}\right]f^{(m)}(x)g^{(n-m+1)}(x)\\ &\quad + f^{(n+1)}(x)g(x)\\ &= \sum_{m=0}^{n+1}\binom{n+1}{m}f^{(m)}(x)g^{(n-m+1)}(x), \quad x \in N,\end{aligned}$$

from Exercise 1.2.19. Setting $x = x_0$ here yields P_{n+1}.

2.3.13. The "proof" breaks down if $g(x) - g(x_0)$ has zeros in every deleted neighborhood of x_0. We first show that if this is so, then $g'(x_0) = 0$. If $\epsilon > 0$ there is a $\delta > 0$ such that $\left|\dfrac{g(x)-g(x_0)}{x-x_0} - g'(x_0)\right| < \epsilon$ if $0 < |x - x_0| < \delta$. Since every deleted neighborhood of x_0 contains a zero of $g(x) - g(x_0)$, this implies that $|g'(x_0)| < \epsilon$. Since ϵ is an arbitrary positive number, this implies that $g'(x_0) = 0$. Therefore, it suffices to show that $h'(x_0) = 0$, as follows. Since f is differentiable at $g(x_0)$, there is a $\delta_1 > 0$ such that

$$\left|\frac{f(u)-f(g(x_0))}{u-g(x_0)} - f'(g(x_0))\right| < 1 \quad \text{if} \quad 0 < |u - g(x_0)| < \delta_1,$$

so

$$\left|\frac{f(u)-f(g(x_0))}{u-g(x_0)}\right| < \left|f'(g(x_0))\right| + 1 = M \quad \text{if} \quad 0 < |u - g(x_0)| < \delta_1. \tag{A}$$

Now let $\epsilon > 0$. Since $g'(x_0) = 0$, there is a $\delta_2 > 0$ such that

$$\left|\frac{g(x)-g(x_0)}{x-x_0}\right| < \epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta_2. \tag{B}$$

Therefore, $|g(x) - g(x_0)| < \epsilon\delta_2$ if $|x - x_0| < \delta$. Now choose $\delta \le \min(\delta_2, \delta_1/\epsilon)$. If $0 < |x - x_0| < \delta$ and $g(x) \ne g(x_0)$, then $0 < |g(x) - g(x_0)| < \epsilon\delta < \delta_1$ (from (B)), and (A) and (B) imply that

$$\left|\frac{h(x)-h(x_0)}{x-x_0}\right| = \left|\frac{f(g(x))-f(g(x))}{g(x)-g(x_0)}\right|\left|\frac{g(x)-g(x_0)}{x-x_0}\right| < M\epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta.$$

Since $h(x) - h(x_0) = f(g(x)) - f(g(x_0)) = 0$ anyway if $g(x) = g(x_0)$, it follows that

$$\left|\frac{h(x)-h(x_0)}{x-x_0}\right| < M\epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta.$$

Hence $h'(x_0) = 0$.

2.3.14. If $y = f(x)$ then

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = \left(\frac{f(x) - f(x_0)}{x - x_0}\right)^{-1},$$

since $g(f(x)) = x$. Since g is continuous and $x = g(y)$, $x \to x_0$ as $y \to y_0$; hence,

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{y - y_0}\right)^{-1} = 1/f'(x_0).$$

2.3.15. **(a)** Suppose that $\epsilon > 0$. Since $f'(a+)$ exists, f is differentiable on an interval (a, b) and there is a $\delta > 0$ such that (A) $|f'(a+) - f'(c)| < \epsilon$ if $a < c < a + \delta$. Since $f'_+(a)$ exists, f is continuous from the right at a. Therefore, the mean value theorem implies that if $a < x < b$, then $\dfrac{f(x) - f(a)}{x - a} = f'(c)$ for some c in (a, x). This and (A) imply that $\left|\dfrac{f(x) - f(a)}{x - a} - f'(c)\right| < \epsilon$ if $a < x < a + \delta$. Therefore, $f'_+(a) = \displaystyle\lim_{x \to a+} \frac{f(x) - f(a)}{x - a} = f'(a+)$.

(b) Let $f(x) = \begin{cases} -1, & x \le 0, \\ 1, & x > 0. \end{cases}$ Then $f'(x) = 0$ if $x > 0$, so $f'(0+) = 0$; however, $f'_+(0)$ does not exist, since $\dfrac{f(x) - f(0)}{x} = \dfrac{1 - (-1)}{x} = \dfrac{2}{x}$ if $x > 0$.

(c) To prove: *If $f'(a+)$ exists and f is continuous from the right at a, then $f'_+(a) = f'(a+)$.* Under these assumptions we can apply the mean value theorem as in **(a)**.

2.3.16. If $|f'(x)| \le M$ for $x \in (a, b)$, then $|f(x_2) - f(x_1)| \le M|x_1 - x_2|$ for x_1, $x_2 \in (a, b)$ (Theorem 2.3.14). Now use Exercise 2.1.38 and the analogous statement concerning $\displaystyle\lim_{x \to x_0+} f(x)$.

2.3.17. The function $g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & a < x \le b, \\ f'_+(a), & x = a, \end{cases}$ is continuous on $[a, b]$, and μ is between $g(a)$ and $g(b)$. Hence, $g(c) = \mu$ and therefore $f(c) - f(a) = \mu(c - a)$ for some c in (a, b) (Theorem 2.2.10).

2.3.18. If $f'_+(a) < \mu < \dfrac{f(b) - f(a)}{b - a}$ then $f(c) - f(a) = \mu(c - a)$ for some c in (a, b), by Exercise 2.3.17. If $\dfrac{f(b) - f(a)}{b - a} < \mu < f'_-(b)$, let $g(x) = \begin{cases} \dfrac{f(b) - f(a)}{b - a}, & a \le x < b, \\ f'_-(b), & x = b, \end{cases}$ which is continuous on $[a, b]$. Since $g(a) < \mu < g(b)$, Theorem 2.2.10 implies that $g(c) = \mu$ and therefore $f(c) - f(b) = \mu(c - b)$ for some c in (a, b).

2.3.19. **(a)** Exercise 2.3.8 implies that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = \frac{\cos 0}{1} = 1$. Therefore, if we define $f(0) = 1$, then f is continuous at 0.

(b) If $x \neq 0$, then $f(x) = \dfrac{\sin x}{x}$ and $f'(x) = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$. Solving these two equations for $\sin x$ and $\cos x$ yields $\sin x = xf$ and $\cos x = xf' + f$. Since $\sin^2 x + \cos^2 x = 1$, $(1+x^2)f^2 + x^2(f')^2 + 2xff' = 1$. Therefore, if $\overline{x} \neq 0$ is a relative extreme point of f, then (A) $|f(\overline{x})| = (1+\overline{x}^2)^{-1/2}$. Notice that this also holds if $\overline{x} = 0$, from **(a)**.

(c) Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, $|f|$ attains a maximum at some $\overline{x}$. Hence **(a)** and **(b)** imply that $|f(x)| \le 1$, with equality if and only if $x = 0$.

2.3.20. **(a)** Since $\sin nk\pi = \sin k\pi = 0$, Exercise 2.3.8, implies that $\lim_{x\to k\pi} \dfrac{\sin nx}{n \sin x} = \dfrac{n\cos nk\pi}{n\cos k\pi} = (-1)^{(n-1)k}$. Therefore, if we define $f(k\pi) = (-1)^{(n-1)k}$, then f is continuous at $k\pi$.

(b) If $x \neq k\pi$ then $f(x) = \dfrac{\sin nx}{n\sin x}$ and $f'(x) = -\dfrac{\cos x \sin nx}{n\sin^2 x} + \dfrac{\cos nx}{\sin x}$. Solving these two equations for $\sin nx$ and $\cos nx$ yields $\sin nx = nf\sin x$, $\cos nx = f'\sin x + f\cos x$. Since $\sin^2 nx + \cos^2 nx = 1$,

$$f^2\left[1 + (n^2-1)\sin^2 x\right] + (f')^2\sin^2 x + 2ff'\sin x\cos x = 1.$$

Therefore, if $\overline{x} \neq 2k\pi$ is a relative extreme point of f, then $|f(\overline{x})| = \left[1 + (n^2-1)\sin^2\overline{x}\right]^{-1/2}$. Notice that this also holds if $\overline{x} = k\pi$, from **(a)**.

(c) Since $f(x+2\pi) = f(x)$, $|f|$ attains its maximum at some $\overline{x}$ in $[0, 2\pi]$. Either $\overline{x} = 0$, $\overline{x} = 2\pi$, or $f'(\overline{x}) = 0$. Hence **(a)** and **(b)** imply that $|f(x)| \le 1$, with equality if and only if $x = k\pi$.

2.3.21. Trivial if $p = 1$. If $p > 1$ let $x_1 < x_2 < \cdots < x_p$. From Rolle's theorem, f' has at least one zero in (x_i, x_{i+1}), $1 \le i \le p-1$. This accounts for at least $p-1$ zeros of f'. In addition, f' has at least $n_i - 1$ zeros, counting multiplicities, at each x_i. Therefore, f' has at least $(n_1-1) + (n_2-1) + \cdots + (n_p-1) + p - 1 = n-1$ zeros, counting multiplicities, in I.

2.3.23. Let $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ be in (a,b) and $y_i < x_i$, $1 \le i \le n$. Show that if f is differentiable on (a,b), then $\sum_{i=1}^{n}[f(x_i) - f(y_i)] = f'(c)\sum_{i=1}^{n}(x_i - y_i)$ for some c in (a,b).

2.3.24. Counterexample: Let $f(x) = \begin{cases} |x|^{3/2}\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ If $x \neq 0$, then $f'(x) = \dfrac{3}{2}\dfrac{|x|^{3/2}}{x}\sin\left(\dfrac{1}{x}\right) - \dfrac{1}{|x|^{1/2}}\cos\left(\dfrac{1}{x}\right)$. Since $f'(0) = \lim_{x\to 0}\dfrac{|x|^{3/2}\sin(1/x)}{x} = 0$, f is differentiable on $(-\infty,\infty)$. However, f does not satisfy a Lipschitz condition at $x_0 = 0$. To see this, we exhibit a set of points $\{x_k\}_{k=1}^{\infty}$ with 0 as a limit point such that $\lim_{k\to\infty}\left|\dfrac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}\right| = \infty$. Let $x_k = 2/(2k+1)\pi$, $k \ge 0$. Then $f(x_k) = (-1)^k x_k^{3/2}$, so

$$\left|\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}\right| = \frac{x_{k+1}^{3/2} + x_k^{3/2}}{x_k - x_{k+1}} > \frac{2x_{k+1}^{3/2}}{x_k - x_{k+1}} = (2k+1)\sqrt{\frac{2}{\pi(2k+3)}},$$

which approaches ∞ as $k \to \infty$.

2.3.25. **(a)** $W' = f'g' + f''g - f'g' - fg'' = -p(fg - fg) = 0$ for all x in (a, b), so Theorem 2.3.12 implies that W is constant on (a, b).

(b) Since $W \neq 0$, $g(x_1) \neq 0$ and $g(x_2) \neq 0$. If g has no zeros in (x_1, x_2), then $h = f/g$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Since $h(x_1) = h(x_2) = 0$, Rolle's theorem implies that $h' = W/g^2$ vanishes somewhere in (x_1, x_2), a contradiction.

2.3.26. $\displaystyle\lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0+} \frac{\sqrt{x}}{x} = \infty$ and $\displaystyle\lim_{x\to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0-} \frac{-\sqrt{-x}}{x} = \infty$, so $f'(0) = \infty$.

2.3.27. Counterexample: $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ and $x_0 = 0$; f is not continuous at 0, but $f'(0) = \displaystyle\lim_{x\to 0} \frac{1}{|x|} = \infty$.

2.3.28. **(a)** The equation of the tangent line to $y = h(x)$ at $(x_0, h(x_0))$ is (A) $y = h(x_0) + h'(x_0)(x - x_0)$. Since $h(x_0) = f(x_0)g(x_0)$ and $h'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) = f(x_0)g'(x_0) \neq 0$, from our assumption, (A) can be rewritten as (B) $y = f(x_0)[g(x_0) + g'(x_0)(x - x_0)]$. The equation of the tangent line to $y = g(x)$ at $(x_0, g(x_0))$ is (C) $y = g(x_0) + g'(x_0)(x - x_0)$. From (B) and (C), both tangent lines intersect the x-axis at $\overline{x} = x_0 - \dfrac{g(x_0)}{g'(x_0)}$.

(b) Apply **(a)** with $g(x) = x - x_1$, so $\overline{x} = x_0 - \dfrac{x_0 - x_1}{1} = x_1$.

(c) Apply **(a)** with $g(x) = (x - x_0)^2$, so $\overline{x} = x_0 - \dfrac{(x_0 - x_1)^2}{2(x_0 - x_1)} = x_0 - \dfrac{(x_0 - x_1)}{2} = \dfrac{x_0 + x_1}{2}$.

(d) Let $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$, so $x_0 = -\dfrac{b}{2a}$ is a critical point of f. However, $f(x_0) = -\dfrac{b^2 - 4ac}{4a} \neq 0$. Now apply **(b)**

(e) The assumptions imply that h is as in **(d)**, where $\gamma = x_1$ and α and β are the zeros of $f(x) = ax^2 + bx + c$, with $a \neq 0$. Since α and β are distinct, $b^2 - 4ac \neq 0$. Therefore, **(d)** implies the conclusion. Assume that f is differentiable on $(-\infty, \infty)$ and x_0 is a critical point of f.

2.4 L'HOSPITAL'S RULE

2.4.1. If $\lim_{x\to b-}\dfrac{f'(x)}{g'(x)}=\infty$ and M is an arbitrary real number, there is an x_0 in (a,b) such that $\dfrac{f'(c)}{g'(c)}>M$ if $x_0<c<b$. By the argument given in the text, we can assume also that g has no zeros in $[x_0,b)$ and (A) $\dfrac{f(x)-f(t)}{g(x)-g(t)}>M$ if $x,t\in[x_0,b)$. If $\lim_{t\to b-}f(t)=\lim_{t\to b-}g(t)=0$ then letting $t\to b-$ in (A) shows that $\dfrac{f(x)}{g(x)}\ge M$ if $x,t\in[x_0,b)$, so $\lim_{x\to b-}\dfrac{f(x)}{g(x)}=\infty$ in this case. If $\lim_{t\to b-}f(t)=\lim_{t\to b-}g(t)=\infty$, let u and x_1 be as in the proof given in the text. Then (B) $\dfrac{f(x)}{g(x)u(x)}>M$ if $x_1<x<b$. Since $\lim_{x\to b-}u(x)=1$, there is an $x_2\ge x_1$ such that $u(x)\ge\dfrac{1}{2}$ if $x_2<x<b$. Therefore, (B) implies that $\dfrac{f(x)}{g(x)}>\dfrac{M}{2}$ if $x_2<x<b$, so $\lim_{x\to b-}\dfrac{f(x)}{g(x)}=\infty$ in this case also.

2.4.2. $\lim_{x\to0}\dfrac{\tan^{-1}x}{\sin^{-1}x}=\lim_{x\to0}\dfrac{1/(1+x^2)}{1/\sqrt{1-x^2}}=1.$

2.4.3. $\lim_{x\to0}\dfrac{1-\cos x}{\log(1+x^2)}=\lim_{x\to0}\dfrac{\sin x}{2x/(1+x^2)}=\left(\lim_{x\to0}\dfrac{1+x^2}{2}\right)\left(\lim_{x\to0}\dfrac{\sin x}{x}\right)=\dfrac{1}{2}.$

2.4.4. $\lim_{x\to0+}\dfrac{1+\cos x}{e^x-1}=\infty$ (not an indeterminate form).

2.4.5. $\lim_{x\to\pi}\dfrac{\sin nx}{\sin x}=\lim_{x\to\pi}\dfrac{n\cos nx}{\cos x}=(-1)^{n-1}n.$

2.4.6. $\lim_{x\to0}\dfrac{\log(1+x)}{x}=\lim_{x\to0}\dfrac{1/(1+x)}{1}=1.$

2.4.7. $\lim_{x\to\infty}e^x\sin e^{-x^2}=\lim_{x\to\infty}\dfrac{\sin e^{-x^2}}{e^{-x}}=\lim_{x\to\infty}\dfrac{-2xe^{-x^2}\cos e^{-x^2}}{-e^{-x}}$
$=2\left(\lim_{x\to\infty}\cos e^{-x^2}\right)\left(\lim_{x\to\infty}\dfrac{x}{e^{x^2-x}}\right)=2\left(\lim_{x\to\infty}\dfrac{1}{(2x-1)e^{x^2-x}}\right)=0.$

2.4.8. $\lim_{x\to\infty}x\sin(1/x)=\lim_{x\to\infty}\dfrac{\sin(1/x)}{1/x}=\lim_{x\to\infty}\dfrac{-(1/x)^2\cos(1/x)}{-(1/x)^2}=\lim_{x\to\infty}\cos(1/x)=$ 1.

2.4.9. $\lim_{x\to\infty}\sqrt{x}(e^{-1/x}-1)=\lim_{x\to\infty}\dfrac{e^{-1/x}-1}{x^{-1/2}}=\lim_{x\to\infty}\dfrac{-(1/x)^2e^{-1/x}}{-x^{-3/2}/2}=2\lim_{x\to\infty}\dfrac{e^{-1/x}}{\sqrt{x}}=$ 0.

2.4.10. $\lim_{x\to0+}\tan x\log x=\lim_{x\to0+}\dfrac{\log x}{\cot x}=\lim_{x\to0+}\dfrac{1/x}{-\csc^2x}=-\lim_{x\to0+}\dfrac{\sin^2x}{x}=$
$-\lim_{x\to0+}\dfrac{2\sin x\cos x}{1}=0.$

2.4.11. $\lim_{x\to\pi} \sin x \log(|\tan x|) = \lim_{x\to\pi} \dfrac{\log(|\tan x|)}{\csc x} = \lim_{x\to\pi} \dfrac{\sec^2 x/\tan x}{-\csc^2 x \cot x} =$
$-\lim_{x\to\pi} \dfrac{\sin x/\cos^3 x}{\cos x/\sin^2 x} = -\lim_{x\to\pi} \tan^4 x = 0.$

2.4.12. $\lim_{x\to 0+} \left[\dfrac{1}{x} + \log(\tan x)\right] = \lim_{x\to 0+} \dfrac{1 + x\log(\tan x)}{x} =$
$\lim_{x\to 0+} \left[\log(\tan x) + x\sec^2 x/\tan x\right] = \lim_{x\to 0+} \log(\tan x) +$
$\left(\lim_{x\to 0+} \sec^3 x\right)\left(\lim_{x\to 0+} \dfrac{x}{\sin x}\right) = -\infty + 1\cdot 1 = -\infty.$

2.4.13. $\lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) = \lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x})\left(\dfrac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\right) =$
$\lim_{x\to\infty} \dfrac{(\sqrt{x+1})^2 - (\sqrt{x})^2}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{x+1}+\sqrt{x}} = 0.$

2.4.14. $\lim_{x\to 0} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x}\right) = \lim_{x\to 0} \dfrac{x - e^x + 1}{x(e^x - 1)} = \lim_{x\to 0} \dfrac{1 - e^x}{e^x - 1 + xe^x} = \lim_{x\to 0} \dfrac{-e^x}{2e^x + x} =$
$-\dfrac{1}{2}.$

2.4.15. $\lim_{x\to 0} (\cot x - \csc x) = \lim_{x\to 0} \dfrac{\cos x - 1}{\sin x} = \lim_{x\to 0} \dfrac{-\sin x}{\cos x} = 0.$

2.4.16. $\lim_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) = \lim_{x\to 0} \dfrac{x - \sin x}{x\sin x} = \lim_{x\to 0} \dfrac{1 - \cos x}{\sin x + x\cos x} =$
$\lim_{x\to 0} \dfrac{\sin x}{2\cos x - x\sin x} = 0.$

2.4.17. $|\sin x|^{\tan x} = \exp[\tan x \log(|\sin x|)]$ and $\lim_{x\to\pi} \tan x \log(|\sin x|) =$
$\lim_{x\to\pi} \dfrac{\log(|\sin x|)}{\cot x} = \lim_{x\to\pi} \dfrac{\cot x}{-\csc^2 x} = -\lim_{x\to\pi} \tan x = 0$, so $\lim_{x\to\pi} |\sin x|^{\tan x} = 1.$

2.4.18. $|\tan x|^{\cos x} = \exp[\cos x \log(|\tan x|)]$ and $\lim_{x\to\pi/2} \cos x \log(|\tan x|) =$
$\lim_{x\to\pi/2} \dfrac{\log(|\tan x|)}{\sec x} = \lim_{x\to\pi/2} \dfrac{\sec^2 x/\tan x}{\sec x \tan x} = \lim_{x\to\pi/2} \dfrac{\cos x}{\sin^2 x} = 0$, so
$\lim_{x\to\pi/2} |\tan x|^{\cos x} = 1.$

2.4.19. $|\sin x|^x = \exp[x\log(|\sin x|)]$ and $\lim_{x\to 0} x\log(|\sin x|) = \lim_{x\to 0} \dfrac{\log(|\sin x|)}{1/x} =$
$\lim_{x\to 0} \dfrac{\cos x/\sin x}{-1/x^2} = -\left(\lim_{x\to 0} \cos x\right)\left(\lim_{x\to 0} \dfrac{x^2}{\sin x}\right) = -\lim_{x\to 0} \dfrac{2x}{\cos x} = 0,$
so $\lim_{x\to 0} |\sin x|^x = 1.$

2.4.20. $(1+x)^{1/x} = \exp\left(\dfrac{\log(1+x)}{x}\right)$ and $\lim_{x\to 0} \dfrac{\log(1+x)}{x} = \lim_{x\to 0} \dfrac{1}{1+x} = 1,$
so $\lim_{x\to 0} (1+x)^{1/x} = e.$

2.4.21. $x^{\sin(1/x)} = \exp[(\log(|x|))\sin(1/x)]$ and $\lim_{x\to\infty}(\log(|x|))\sin(1/x) =$
$-\lim_{t\to 0}(\log(|t|))\sin t = -\lim_{t\to 0}\dfrac{\log(|t|)}{1/\sin t} = \lim_{t\to 0}\dfrac{1/t}{\cos t/\sin^2 t} =$
$\left(\lim_{t\to 0}\dfrac{1}{\cos t}\right)\left(\lim_{t\to 0}\dfrac{\sin^2 t}{t}\right) = \lim_{t\to 0} 2\sin t\cos t = 0$, so $\lim_{x\to\infty} x^{\sin(1/x)} = 1$.

2.4.22. $\lim_{x\to 0}\left(\dfrac{x}{1-\cos x} - \dfrac{2}{x}\right) = \lim_{x\to 0}\dfrac{x^2-2+2\cos x}{x(1-\cos x)} = 2\lim_{x\to 0}\dfrac{x-\sin x}{1-\cos x + x\sin x} =$
$2\lim_{x\to 0}\dfrac{1-\cos x}{2\sin x + x\cos x} = 2\lim_{x\to 0}\dfrac{\sin x}{3\cos x - x\sin x} = 0.$

2.4.23. If $\alpha < 0$, then $\lim_{x\to 0+} x^\alpha \log x = \left(\lim_{x\to 0+} x^\alpha\right)\left(\lim_{x\to 0+}\log x\right) = \infty(-\infty) = -\infty.$

If $\alpha = 0$, then $\lim_{x\to 0+} x^\alpha \log x = \left(\lim_{x\to 0+}\log x\right) = -\infty.$

If $\alpha > 0$, then $\lim_{x\to 0+} x^\alpha \log x = \lim_{x\to 0+}\dfrac{\log x}{1/x^\alpha} = \lim_{x\to 0+}\dfrac{1/x}{-\alpha/x^{\alpha+1}} = -\dfrac{1}{\alpha}\lim_{x\to 0+} x^\alpha = 0.$

2.4.24. $\lim_{x\to e}\dfrac{\log(\log x)}{\sin(x-e)} = \lim_{x\to e}\dfrac{1/(x\log x)}{\cos(x-e)} = \dfrac{1}{e}$

2.4.25. $\left(\dfrac{x+1}{x-1}\right)^{\sqrt{x^2-1}} = \exp\left[\sqrt{x^2-1}\,\log\left(\dfrac{x+1}{x-1}\right)\right]$ and $\lim_{x\to\infty}\sqrt{x^2-1}\,\log\left(\dfrac{x+1}{x-1}\right) =$
$\lim_{x\to\infty}\dfrac{\log\left(\dfrac{x+1}{x-1}\right)}{1/\sqrt{x^2-1}} = \lim_{x\to\infty}\dfrac{-2/(x^2-1)}{-x/(x^2-1)^{3/2}} = 2\lim_{x\to\infty}\dfrac{\sqrt{x^2-1}}{x} = \lim_{x\to\infty}\sqrt{1-1/x^2}$
$= 2$, so $\lim_{x\to\infty}\left(\dfrac{x+1}{x-1}\right)^{\sqrt{x^2-1}} = e^2.$

2.4.26. $\left(\dfrac{x+1}{x-1}\right)^{\sqrt{x^2-1}} = \exp\left[\sqrt{x^2-1}\,\log\left(\dfrac{x+1}{x-1}\right)\right]$ and $\lim_{x\to 1+}\sqrt{x^2-1}\,\log\left(\dfrac{x+1}{x-1}\right) =$
$\lim_{x\to 1+}\dfrac{\log\left(\dfrac{x+1}{x-1}\right)}{1/\sqrt{x^2-1}} = \lim_{x\to 1+}\dfrac{-2/(x^2-1)}{-x/(x^2-1)^{3/2}} = 2\lim_{x\to 1+}\dfrac{\sqrt{x^2-1}}{x} = \lim_{x\to 1+}\sqrt{1-1/x^2}$
$= 0$, so $\lim_{x\to 1+}\left(\dfrac{x+1}{x-1}\right)^{\sqrt{x^2-1}} = 1.$

2.4.27. If $\beta < 0$, then $\lim_{x\to\infty}\dfrac{(\log x)^\beta}{x} = \left(\lim_{x\to\infty}(\log x)^\beta\right)\left(\lim_{x\to\infty}\dfrac{1}{x}\right) = 0\cdot 0 = 0$. If $\beta = 0$, then $\lim_{x\to\infty}\dfrac{(\log x)^\beta}{x} = \left(\lim_{x\to\infty}\dfrac{1}{x}\right) = 0$. If $\beta > 0$ and k is the smallest positive integer such that $\beta \le k$, then k applications of L'Hospital's rule yields $\lim_{x\to\infty}\dfrac{(\log x)^\beta}{x} =$

$\beta \lim_{x\to\infty} \frac{(\log x)^{\beta-1}}{x} = \cdots = \beta(\beta-1)\cdots(\beta-k+1) \lim_{x\to\infty} \frac{(\log x)^{\beta-k}}{x}$
$= 0.$

2.4.28. $\lim_{x\to\infty} (\cosh x - \sinh x) = \lim_{x\to\infty} \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \right) = \lim_{x\to\infty} e^{-x} = 0.$

2.4.29. If $\alpha < 0$, then $\lim_{x\to\infty} (x^\alpha - \log x) = \left(\lim_{x\to\infty} x^\alpha \right) - \left(\lim_{x\to\infty} \log x \right) = 0 - \infty = -\infty$. If $\alpha = 0$, then $\lim_{x\to\infty} (x^\alpha - \log x) = \left(\lim_{x\to\infty} 1 \right) - \left(\lim_{x\to\infty} \log x \right) = 1 - \infty = -\infty$. Suppose that $\alpha > 0$. Then (A) $\lim_{x\to\infty} (x^\alpha - \log x) = \left(\lim_{x\to\infty} x^\alpha \right) \left(\lim_{x\to\infty} (1 - x^{-\alpha} \log x) \right)$. Since $\lim_{x\to\infty} x^\alpha = \infty$ and $\lim_{x\to\infty} \frac{\log x}{x^\alpha} = \frac{1}{\alpha} \lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \frac{1}{\alpha} \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0$, (A) implies that $\lim_{x\to\infty} (x^\alpha - \log x) = \infty \cdot 1 = \infty$.

2.4.30. $\lim_{x\to-\infty} e^{x^2} \sin(e^x) = \lim_{x\to-\infty} \frac{\sin(e^x)}{e^{-x^2}} = \lim_{x\to-\infty} \frac{e^x \cos(e^x)}{-2xe^{-x^2}} =$
$-\frac{1}{2} \left(\lim_{x\to-\infty} \cos(e^x) \right) \left(\lim_{x\to-\infty} \frac{e^{x^2+x}}{x} \right) = -\frac{1}{2} \lim_{x\to-\infty} (2x+1)e^{x^2+x} = \infty.$

2.4.31. $\lim_{x\to\infty} x(x+1) [\log(1+1/x)]^2 = \lim_{x\to\infty} \frac{[\log(1+1/x)]^2}{1/x(x+1)}$
$= \lim_{x\to\infty} \left(\frac{-2\log(1+1/x)}{x(x+1)} \right) \Big/ \left(\frac{-(2x+1)}{x^2(x+1)^2} \right) = \lim_{x\to\infty} \frac{2x(x+1)\log(1+1/x)}{2x+1} =$
$\left(\lim_{x\to\infty} \frac{2x+2}{2x+1} \right) \left(\lim_{x\to\infty} x \log(1+1/x) \right) = \left(\lim_{x\to\infty} \frac{2}{2} \right) \left(\lim_{x\to\infty} \frac{\log(1+1/x)}{1/x} \right) =$
$\lim_{x\to\infty} \frac{(-1/x)^2/(1+1/x)}{(-1/x)^2} = 1.$

2.4.32. $\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5} = \lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{5x^4} = \lim_{x\to 0} \frac{-\sin x + x}{20x^3} =$
$\lim_{x\to 0} \frac{-\cos x + 1}{60x^2} = \lim_{x\to 0} \frac{\sin x}{120x} = \frac{1}{120}.$

2.4.33. If $\alpha < 0$, then $\lim_{x\to\infty} \frac{e^x}{x^\alpha} = \left(\lim_{x\to\infty} \frac{1}{x^\alpha} \right) \left(\lim_{x\to\infty} e^x \right) = \infty \cdot \infty = \infty$. If $\alpha = 0$, then $\lim_{x\to\infty} \frac{e^x}{x^\alpha} = \left(\lim_{x\to\infty} e^x \right) = \infty$. If $\alpha > 0$ and k is the smallest positive integer such that $\alpha \le k$, then k applications of L'Hospital's rule yields $\lim_{x\to\infty} \frac{e^x}{x^\alpha} =$
$\frac{1}{\alpha} \lim_{x\to\infty} \frac{e^x}{x^{\alpha-1}} = \cdots = \frac{1}{\alpha(\alpha-1)\cdots(\alpha-k+1)} \lim_{x\to\infty} \frac{e^x}{x^{\alpha-k}} = \infty.$

2.4.34. $\lim_{x\to 3\pi/2-} e^{\tan x} \cos x = \lim_{x\to 3\pi/2-} \frac{e^{\tan x}}{1/\cos x} = \lim_{x\to 3\pi/2-} \frac{\sec^2 x e^{\tan x}}{\sin x \sec^2 x} =$
$\lim_{x\to 3\pi/2-} \frac{e^{\tan x}}{\sin x} == -\infty.$

2.4.35. If $\alpha < 0$, then $\lim_{x\to 1+}(\log x)^\alpha \log(\log x) = \left(\lim_{x\to 1+}(\log x)^\alpha\right)\left(\lim_{x\to 1+}\log(\log x)\right) = \infty(-\infty) = -\infty$. If $\alpha = 0$, then $\lim_{x\to 1+}(\log x)^\alpha \log(\log x) = \left(\lim_{x\to 1+}\log(\log x)\right) = -\infty$. If $\alpha > 0$, then $\lim_{x\to 1+}(\log x)^\alpha \log(\log x) = \lim_{x\to 1+}\dfrac{\log(\log x)}{(\log x)^{-\alpha}} = \lim_{x\to 1+}\dfrac{(\log x)/x}{-\alpha(\log x)^{-\alpha-1}/x} = -\dfrac{1}{\alpha}\lim_{x\to 1+}(\log x)^{\alpha+2} = 0$.

2.4.36. $\lim_{x\to\infty}\dfrac{x^x}{x\log x} = \lim_{x\to\infty}\dfrac{x^x(x\log x)'}{(x\log x)'} = \lim_{x\to\infty}x^x = \infty$.

2.4.37. $(\sin x)^{\tan x} = \exp[\tan x\log(\sin x)]$ and $\lim_{x\to\pi/2}\tan x\log(\sin x) = \lim_{x\to\pi/2}\dfrac{\log(\sin x)}{\cot x} = \lim_{x\to\pi/2}\dfrac{\cot x}{-\csc^2 x} = \lim_{x\to\pi/2}\sin x\cos x = 0$, so $\lim_{x\to\pi/2}(\sin x)^{\tan x} = 1$.

2.4.38. We prove by induction that $\lim_{x\to 0}\dfrac{e^x - \sum_{r=0}^{n}\dfrac{x^r}{r!}}{x^n} = 0$ if $n \ge 1$. We first verify P_1: $\lim_{x\to 0}\dfrac{e^x - 1 - x}{x} = \lim_{x\to 0}\dfrac{e^x - 1}{1} = 0$. Now suppose that $n \ge 1$ and P_n is true. Applying L'Hospital's rule yields

$$\lim_{x\to 0}\frac{e^x - \sum_{r=0}^{n+1}\frac{x^r}{r!}}{x^{n+1}} = \lim_{x\to 0}\frac{e^x - \sum_{r=1}^{n+1}\frac{x^{r-1}}{(r-1)!}}{(n+1)x^n} = \lim_{x\to 0}\frac{e^x - \sum_{r=0}^{n}\frac{x^r}{r!}}{(n+1)x^n} = 0,$$

by P_n. Therefore, P_n implies P_{n+1}.

2.4.39. We prove by induction that $\lim_{x\to 0}\dfrac{\sin x - \sum_{r=0}^{n}(-1)^r\dfrac{x^{2r+1}}{(2r+1)!}}{x^{2n+1}} = 0$ if $n \ge 0$. We first verify P_0: $\lim_{x\to 0}\dfrac{\sin x - x}{x} = \lim_{x\to 0}\dfrac{\cos x - 1}{1} = 0$. Now suppose that $n \ge 0$ and P_n is true. Applying L'Hospital's rule twice yields

$$\lim_{x\to 0}\frac{\sin x-\displaystyle\sum_{r=0}^{n+1}(-1)^r\frac{x^{2r+1}}{(2r+1)!}}{x^{2n+3}}=\lim_{x\to 0}\frac{\cos x-\displaystyle\sum_{r=0}^{n+1}(-1)^r\frac{x^{2r}}{(2r)!}}{(2n+3)x^{2n+2}}$$

$$=\lim_{x\to 0}\frac{-\sin x-\displaystyle\sum_{r=1}^{n+1}(-1)^r\frac{x^{2r-1}}{(2r-1)!}}{(2n+2)(2n+3)x^{2n+1}}$$

$$=-\lim_{x\to 0}\frac{\sin x-\displaystyle\sum_{r=0}^{n}(-1)^r\frac{x^{2r+1}}{(2r+1)!}}{(2n+2)(2n+3)x^{2n+1}}=0,$$

by P_n. Therefore, P_n implies P_{n+1}.

2.4.40. (Proof by induction.) P_k is obvious if $k \le 0$. Suppose that $n \ge 0$ and P_k is true for $k \le n$. Then

$$\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{n+1}}=\lim_{x\to 0}\frac{x^{-n-1}}{e^{1/x^2}}=\lim_{x\to 0}\frac{-(n+1)x^{-n-2}}{-2e^{1/x^2}/x^3}=\frac{n+1}{2}\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{n-1}}=0.$$

Hence, P_{n+1} is true.

2.4.41. **(a)** Since f is continuous at x_0, $\lim_{x\to x_0}(f(x)-f(x_0))=0$. Since $\lim_{x\to x_0}(x-x_0)=0$ and $\lim_{x\to x_0} f'(x)$ exists, L'Hospital's rule implies that $\displaystyle\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ $=\lim_{x\to x_0} f'(x)$. Therefore, $f'(x_0)$ exists and equals $\lim_{x\to x_0} f'(x)$, so f' is continuous at x_0.

(b) Let g' be continuous on $(-\infty,\infty)$ and define $f(x)=\begin{cases} g(x), & x\le x_0,\\ 1+g(x), & x>x_0.\end{cases}$ The $f'(x)=g'(x)$ if $x\ne x_0$, so $\lim_{x\to x_0} f'(x)=\lim_{x\to x_0} g'(x)$ exists. However, f is not continuous at x_0, so $f'(x_0)$ does not exist.

2.4.42. **(a)** (Proof by induction.) Since $L_1(x)=\log(L_0(x))=\log x=L_0(\log x)$, P_1 is true. Now suppose that $n\ge 1$ and P_n is true. Then $L_{n+1}(x)=\log(L_n(x))=\log(L_{n-1}(\log x))$ (by P_n) $=L_{n-1}(\log x)$. Hence, P_n implies P_{n+1}.

(b) (Proof by induction.) Since $L_0(0+)=0$ and $L_1(0+)=\log(0+)=-\infty$, P_1 is true. Now suppose that $n\ge 1$ and P_n is true. Then $L_n(a_{n+1}+)=L_{n-1}(\log(a_{n+1})+)=L_{n-1}(\log(e^{a_n})+)=L_{n-1}(a_n+)=0$ and $L_{n+1}(a_{n+1}+)=L_n(\log(a_{n+1})+)=L_n(\log(e^{a_n})+)$ $=L_n(a_n+)=-\infty$, so P_{n+1} is true.

(c) Since $L_{n-1}(a_n+)=0$ and $L_n(a_n+)=-\infty$, L'Hospital's rule yields $\displaystyle\lim_{x\to a_n+}(L_{n-1}(x))^\alpha L_n(x)=\lim_{x\to a_n+}\frac{L_n(x)}{(L_{n-1}(x))^{-\alpha}}=\lim_{x\to a_n+}\frac{L'_n(x)}{-\alpha(L_{n-1}(x))^{-\alpha-1}L'_{n-1}(x)}=$ $\displaystyle-\frac{1}{\alpha}\lim_{x\to a_n+}(L_{n-1}(x))^\alpha$ since $L'_n=L'_{n-1}/L_{n-1}$. Thus, $\displaystyle\lim_{x\to a_n+}(L_{n-1}(x))^\alpha L_n(x)=(L_{n-1}(a_n)+)^\alpha=$ 0, from **(b)**.

(d) We first prove by induction that (A) $\lim_{x\to\infty} L_n(x) = \infty$. This is true for $n = 0$. Now suppose that $n \ge 0$ and (A) is true. Since $L_{n+1}(x) = \log(L_n(x))$, (A) implies that $\lim_{x\to\infty} L_{n+1}(x) = \infty$ (Exercise 2.1.25). If $\alpha < 0$, then $\lim_{x\to\infty} (L_n(x))^\alpha / L_{n-1}(x) = \left(\lim_{x\to\infty} (L_n(x))^\alpha\right)\left(\lim_{x\to\infty} 1/L_{n-1}(x)\right) = 0{\cdot}0 = 0$. If $\alpha = 0$, then $\lim_{x\to\infty} (L_n(x))^\alpha / L_{n-1}(x) = \left(\lim_{x\to\infty} 1/L_{n-1}(x)\right) = 0$. If $\alpha > 0$, let k be the smallest integer such that $\alpha \le k$. Then k applications of L'Hospital's rule (using $L_n' = L_{n-1}'/L_{n-1}$ each time) yields

$$\lim_{x\to\infty} (L_n(x))^\alpha / L_{n-1}(x) = \alpha(\alpha-1)\dots(\alpha-k+1) \lim_{x\to\infty} (L_n(x))^{\alpha-k} / L_{n-1}(x) = 0.$$

2.4.43. If $0 < L_1 < L$, there is an x_0 such that $f'(x) > L_1 f(x) >$ if $x \ge x_0$. Since $f(x) > 0$ on $(0,\infty)$, it follows that $f'(x) > 0$ if $x \ge x_0$, so $f(x) > f(x_0)$ if $x > x_0$ and $f'(x) > Lf(x_0)$ if $x > x_0$. Therefore, $f(x) > f(x_0)\,(1 + L_1(x - x_0))$ if $x > x_0$ (Theorem 2.3.11), so $\lim_{x\to\infty} f(x) = \infty$. By an induction proof based on Exercise 2.1.25, $\lim_{x\to\infty} f_n(x) = \infty$. Since (A) $f_n'(x) = f'(f_{n-1}(x)) f_{n-1}'(x)$, an induction proof shows that $f_n'(x) > 0$ for sufficiently large x. Hence, L'Hospital's rule yields $\lim_{x\to\infty} \dfrac{(f_n(x))^\alpha}{f_{n-1}(x)} = \alpha \lim_{x\to\infty} \dfrac{(f_n(x))^{\alpha-1} f_n'(x)}{f_{n-1}'(x)}$ if the limit on the right exists in the extended reals. Because of (A) this implies that (B) $\lim_{x\to\infty} \dfrac{(f_n(x))^\alpha}{f_{n-1}(x)} = \alpha \lim_{x\to\infty} (f_n(x))^\alpha f'(f_{n-1}(x))$ if the limit on the right exists in the extended reals. Since $f_n(x) = f(f_{n-1}(x))$, (B) can be rewritten as

$$\lim_{x\to\infty} \frac{(f_n(x))^\alpha}{f_{n-1}(x)} = \alpha \lim_{x\to\infty} (f_n(x))^\alpha \frac{f'(f_{n-1}(x)))}{f(f_{n-1}(x))}. \qquad \text{(C)}$$

However, since $\lim_{x\to\infty} \dfrac{f'(x)}{f(x)} = \infty$ and $\lim_{x\to\infty} f_{n-1}(x) = \infty$, Exercise 2.1.25 implies that $\lim_{x\to\infty} \dfrac{f'(f_{n-1}(x)))}{f(f_{n-1}(x))} = L$. Since $\lim_{x\to\infty} (f_n(x))^\alpha = 0$, (C) implies that $\lim_{x\to\infty} \dfrac{(f_n(x))^\alpha}{f_{n-1}(x)} = \alpha(\infty)L = \infty$.

2.4.44. **(a)** $|f(x)|^{f(x)} = \exp(f(x)) \log|f(x)|$ and $\lim_{x\to x_0} f(x)) \log|f(x)| = \lim_{x\to x_0} \dfrac{\log|f(x)|}{1/f(x)} = \lim_{x\to x_0} \dfrac{f'(x)/f(x)}{-f'(x)/(f(x))^2} = \lim_{x\to x_0} f(x) = 0$, so $\lim_{x\to x_0} |f(x)|^{f(x)} = 1$.

(b) $|f(x)|^{1/(f(x)-1)} = \exp\left(\dfrac{\log|f(x)|}{f(x)-1}\right)$ and $\lim_{x\to x_0} \dfrac{\log|f(x)|}{f(x)-1} = \lim_{x\to x_0} \dfrac{f'(x)/f(x)}{f'(x)} = \lim_{x\to x_0} \dfrac{1}{f(x)} = 1$, so $\lim_{x\to x_0} |f(x)|^{1/(f(x)-1)} = e$.

(c) $|f(x)|^{1/f(x)} = \exp\left(\dfrac{\log|f(x)|}{f(x)}\right)$ and $\lim_{x\to x_0} \dfrac{\log|f(x)|}{f(x)} = \lim_{x\to x_0} \dfrac{f'(x)/f(x)}{f'(x)} = \lim_{x\to x_0} \dfrac{1}{f(x)} = 0$, so $\lim_{x\to x_0} |f(x)|^{1/(f(x)-1)} = 1$.

2.4.45. $(1+f(x))^{1/g(x)} = \exp\left(\dfrac{\log(1+f(x))}{g(x)}\right)$ and $\lim_{x\to b-} \dfrac{\log(1+f(x))}{g(x)} = \lim_{x\to b-} \dfrac{f'(x)}{g'(x)} =$

L, so $\lim_{x\to b-}(1+f(x))^{1/g(x)} = e^L$.

2.4.46. The first four forms are not indeterminate; the rest are. A function may be of the form $\infty - \infty$ as $x \to x_0+$ and $-\infty + \infty$ as $x \to x_0-$, but approach a limit as $x \to x_0$. (For example, $f(x) = \dfrac{1}{x} - \dfrac{\sin x}{x^2}$ with $x_0 = 0$.) Similar comments apply to the other pairs of indeterminate forms.

2.5 TAYLOR'S THEOREM

2.5.1. We show by induction that (A) $f^{(n)}(x) = \begin{cases} q_n(1/x)e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$, where q_n is a polynomial. If $n = 0$, the definition of f implies (A) with $q_0(u) = 1$. Suppose that $n \geq 0$ and (A) is true. If $x \neq 0$, then $f^{(n+1)(x)} = q_{n+1}(1/x)$, with $q_{n+1}(u) = 2u^3 q_n(u) - u^2 q_n'(u)$. Since $f^{(n)}(0) = 0$ (from (A)), $f^{(n+1)}(0) = \lim_{x\to 0} \dfrac{f^{(n)}(x)}{x} = \lim_{x\to 0} \dfrac{q_n(1/x)e^{-1/x^2}}{x}$. However, the last limit equals zero because $\lim_{x\to 0} \dfrac{e^{-1/x^2}}{x^k} = 0$ for every integer k (Exercise 2.4l.40). Since $f^{(n)} = 0$ for $n \geq 0$, every Taylor polynomial of f about 0 is identically zero.

2.5.2. From Theorem 2.5.1, (A) $\lim_{x\to x_0} \dfrac{f(x) - T_{n+1}(x)}{(x-x_0)^{n+1}} = 0$. Since $T_{n+1}(x) = T_n(x) + \dfrac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1}$, (A) implies that $\lim_{x\to x_0} \dfrac{f(x) - T_n(x)}{(x-x_0)^{n+1}} = \dfrac{f^{(n+1)}(x_0)}{(n+1)!}$, which can be rewritten as $\lim_{x\to x_0} \dfrac{E_n(x) - E_n(x_0)}{x - x_0} = \dfrac{f^{(n+1)}(x_0)}{(n+1)!}$. Therefore, $E_n'(x_0) = \dfrac{f^{(n+1)}(x_0)}{(n+1)!}$.

2.5.3. **(a)** The hypotheses implies that (A) $f(x) = a_0 + a_1(x-x_0) + E(x)(x-x_0)$, where (B) $\lim_{x\to x_0} E(x) = 0$. Therefore, $\lim_{x\to x_0} f(x) = a_0$, so $a_0 = f(x)$ because f is continuous at x_0. Now (A) and (B) imply that $f'(x_0) = \lim_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = a_1$.

(b) Let $x_0 = a_0 = a_1 = a_2 = 0$ and $f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0, \\ 0, & x = 0. \end{cases}$ Then f and $f' \begin{cases} 3x^2 \sin 1/x - x\cos 1/x, & x \neq 0, \\ 0, & x = 0, \end{cases}$ are both continuous at 0 and $\lim_{x\to x_0} \dfrac{x^3 \sin 1/x}{x^2} = 0$, but $f''(0)$ does not exist.

2.5.4. **(a)** From Lemma 2.5.2, $f(x_0+h) = f(x_0) + f'(x_0)h + \left(\dfrac{f''(x_0)}{2} + E(x_0+h)\right)h^2$ and $f(x_0-h) = f(x_0) - f'(x_0)h + \left(\dfrac{f''(x_0)}{2} + E(x_0-h)\right)h^2$, where $\lim_{x\to x_0} E_2(x) = 0$, so the limit in question equals $f''(x_0) + \lim_{h\to 0}(E_2(x_0+h) + E_2(x_0-h)) = f''(x_0)$.

(b) Counterexample: Let $x_0 = 0$ and $f(x) = x|x|$. Then

$$\lim_{h\to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} = \lim_{h\to 0} \frac{h|h| - h|h|}{h^2} = 0.$$

However, $f'(x) = 2|x|$, so $f''(0)$ does not exist.

2.5.5. **(a)** For necessity, suppose that f has a simple zero at x_0. Then Lemma 2.3.2 implies that $f(x) = g(x)(x - x_0)$ with $g(x) = f'(x_0) + E_1(x)$, where $\lim_{x\to x_0} E_1(x) = E_1(0) = 0$, so g is continuous at x_0 and $\lim_{x\to x_0} g(x) = f'(x_0) \neq 0$. Since $g(x) = \dfrac{f(x)}{x - x_0}$, g is differentiable on a deleted neighborhood of x_0. For sufficiency, suppose that $f(x) = g(x)(x - x_0)$ where g has the stated properties. Then $f(x_0) = 0$ and $f'(x_0) = \lim_{x\to 0} \dfrac{f(x)}{x - x_0} = \lim_{x\to x_0} g(x) = g(x_0) \neq 0$.

(b) Let $g(x) = 1 + |x - x_0|$, so $f(x) = (x - x_0)(1 + |x - x_0|)$. Then $f(x_0) = 0$ and $f'(x_0) = 1$, but g is not differentiable at x_0.

2.5.6. **(a)** For necessity, suppose that f has a double zero at x_0. Then Lemma 2.5.2 implies that (A) $f(x) = g(x)(x - x_0)^2$ with $g(x) = \dfrac{f'(x_0)}{2} + E_0(x)$, where $\lim_{x\to x_0} E_0(x) = E_0(0) = 0$, so g is continuous at x_0 and (B) $\lim_{x\to x_0} g(x) = \dfrac{f'(x_0)}{2} \neq 0$. Since $g(x) = \dfrac{f(x)}{(x - x_0)^2}$, g is twice differentiable on a deleted neighborhood N of x_0. Differentiating (A) yields $f'(x) = g'(x)(x - x_0)^2 + 2g(x)(x - x_0),\ x \in N$, so (C) $g'(x)(x - x_0) = \dfrac{f'(x)}{x - x_0} - 2g(x)$. Applying Lemma 2.5.2 to f' shows that $\dfrac{f'(x)}{x - x_0} = f''(x_0) + E_1(x)$, where $\lim_{x\to x_0} E_1(x) = 0$; therefore, $\lim_{x\to x_0} \dfrac{f'(x)}{(x - x_0)^2} = f''(x_0)$. This, (B), and (C) imply that $\lim_{x\to x_0} (x - x_0)g'(x) = 0$.

For sufficiency, suppose that $f(x) = g(x)(x - x_0)^2$ where g has the stated properties on a deleted neighborhood N of x_0. Then $f(x_0) = 0$ and $f'(x_0) = \lim_{x\to 0} \dfrac{f(x)}{x - x_0} = \lim_{x\to x_0} g(x)(x - x_0) = 0$. If $x \in N - \{x_0\}$, then $f'(x) = g'(x)(x - x_0)^2 + 2g(x)(x - x_0)$, so $f''(0) = \lim_{x\to x_0} g'(x)(x - x_0) + 2g(x) = 2g(x_0) \neq 0$.

(b) Let $g(x) = 1 + |x - x_0|$, so $f(x) = (x - x_0)^2(1 + |x - x_0|)$. Then $f(x_0) = f'(x_0) = 0$ and $f''(x_0) = 2$, but g is not differentiable at x_0.

2.5.7. Let P_n be the stated proposition. Exercise 2.5.6 implies P_1. We show that if P_n is true for some $n \geq 1$, then P_{n+1} is true..

For necessity, suppose that f has a zero of multiplicity $n + 1$ at x_0. Then Lemma 2.5.2 implies that (A) $f(x) = g(x)(x - x_0)^{n+1}$ with $g(x) = \dfrac{f^{(n+1)}(x_0)}{(n + 1)!} + E_0(x)$, where $\lim_{x\to x_0} E_0(x) = E_0(0) = 0$, so g is continuous at x_0 and (B) $\lim_{x\to x_0} g(x) = \dfrac{f^{(n+1)}(x_0)}{(n + 1)!} \neq 0$. Since $g(x) = \dfrac{f(x)}{(x - x_0)^{n+1}}$, g is $n + 1$ times differentiable on a deleted neighborhood N of x_0. Differentiating (A) yields $f'(x) = g_1(x)(x - x_0)^n,\ x \in N$, where (C) $g_1(x) =$

$g'(x)(x-x_0)+(n+1)g(x)$. However, applying Lemma 2.5.2 to f' shows that (D) $g_1(x) = \dfrac{f^{(n+1)}(x_0)}{n!} + E_1(x)$, where (E) $\lim_{x\to x_0} E_1(x) = 0$. From (C) and (D), $g'(x)(x-x_0) = \dfrac{f^{(n+1)}(x_0)}{n!} + E_1(x) - (n+1)g(x)$. Now (B) and (E) imply that (F) $\lim_{x\to x_0}(x-x_0)g'(x) = 0$. We must still show that (G) $\lim_{x\to x_0}(x-x_0)^j g^{(j)}(x) = 0, 2 \le j \le n$. Since f' has a zero of multiplicity n at x_0, P_n implies that $\lim_{x\to x_0}(x-x_0)^j g_1^{(j)}(x) = 0,\ 1 \le j \le n-1$. From (C) this is equivalent to

$$\lim_{x\to x_0} g^{(j+1)}(x)(x-x_0)^{j+1} + \lim_{x\to x_0} g^{(j)}(x)(x-x_0)^j = 0, \quad 2 \le j \le n-1. \qquad \text{(H)}$$

From (C), (F) and (H) with $j = 1$ imply (G) with $j = 2$. Moreover, if $3 \le r \le n-1$, (G) and (H) with $j = r-1$ imply (G) with $j = r$. This completes the necessity part of P_{n+1}. For sufficiency, suppose that $f(x) = g(x)(x-x_0)^{n+1}$ where g is continuous at x_0 and $n+1$ times differentiable on a deleted neighborhood N of x_0, $g(x_0) \ne 0$, and (H) $\lim_{x\to x_0}(x-x_0)^j g^{(j)}(x) = 0,\ 1 \le j \le n$. Then $f(x_0) = 0$ and $f'(x_0) = \lim_{x\to 0}\dfrac{f(x)}{x-x_0} = \lim_{x\to x_0} g(x)(x-x_0) = 0$. Now define

$$g_1(x) = \begin{cases} g'(x)(x-x_0) + (n+1)g(x), & x \ne 0, \\ (n+1)g(x_0), & x = 0. \end{cases}$$

Then g_1 is continuous at x_0 (since $\lim_{x\to x_0}(x-x_0)g(x) = 0$) and n times differentiable on N, $g_1(x_0) \ne 0$, and $\lim_{x\to x_0}(x-x_0)^j g_1^{(j)}(x) = 0,\ 1 \le j \le n-1$. Since $f'(x) = g_1(x)(x-x_0)^n$, P_n implies that f' has a zero of multiplity n at x_0. Therefore, f has a zero of multiplicity $n+1$ at x_0. This completes the proof of the sufficiency part of P_{n+1}.

2.5.8. **(a)** The assumption implies that (A) $\alpha_0 + \alpha_1(x-x_0) + \cdots + \alpha_n(x-x_0)^n = \phi(x)(x-x_0)^n$, where $\lim_{x\to x_0}\phi(x) = 0$. Letting $x \to x_0$ shows that $\alpha_0 = 0$. Now (A) implies that $\alpha_1 + \alpha_2(x-x_0)\cdots + \alpha_n(x-x_0)^{n-1} = \phi(x)(x-x_0)^{n-1}$. Letting $x \to 0$ here shows that $\alpha_1 = 0$. Applying this argument $n+1$ times yields the conclusion.

(b) By Theorem 2.5.1 $\lim_{x\to x_0}\dfrac{f(x)-T_n(x)}{(x-x_0)^n} = 0$. This and our assumption on p imply that $\lim_{x\to x_0}\dfrac{p(x)-T_n(x)}{(x-x_0)^n} = 0$. Now apply **(a)** with $Q = p - T_n$

2.5.9. Let T_n and S_n be the nth Taylor polynomials of f and g about x_0. Then $\lim_{x\to x_0}\dfrac{T_n(x)-S_n(x)}{(x-x_0)^n} = \lim_{x\to x_0}\dfrac{T_n(x)-f(x)}{(x-x_0)^n} + \lim_{x\to x_0}\dfrac{f(x)-g(x)}{(x-x_0)^n} + \lim_{x\to x_0}\dfrac{g(x)-S_n(x)}{(x-x_0)^n}$.
The first and last limits on the right vanish by Theorem 2.5.1, the second by assumption. Now apply Exercise 2.5.8**(a)** with $Q = T_n - S_n$.

2.5.10. **(a)** $F_nG_n - fg = (F_n - f)G_n + f(G_n - g)$, so (A) $\lim_{x\to x_0}\dfrac{F_n(x)G_n(x) - f(x)g(x))}{(x-x_0)^n} =$

$G_n(x_0)\lim_{x\to x_0}\dfrac{F_n(x)-f(x)}{(x-x_0)^n} + f(x_0)\lim_{x-x_0}\dfrac{G_n(x)-g(x)}{(x-x_0)^n} = 0$, by Theorem 2.5.1. Now let P_n be the polynomial obtained by retaining only the powers of $x-x_0$ through the nth in F_nG_n. Then $P_n = F_nG_n - a_{n+1}(x-x_0)^{n+1} + \cdots + a_{2n}(x-x_0)^{2n}$, where $a_{n+1}, \dots, a_{2n}$ are constants. Since $\lim_{x\to x_0}\dfrac{(x-x_0)^{n+k}}{(x-x_0)^n} = 0$ if $k > 0$, (A) implies that $\lim_{x\to x_0}\dfrac{P_n(x)-f(x)g(x))}{(x-x_0)^n} = 0$. Therefore, Exercise 2.5.8(**b**) implies that $P_n = H_n$.

(b) **(i)** Let $f(x) = e^x$, $g(x) = \sin x$. Then $F_4(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$ and $G_4(x) = x - \dfrac{x^3}{6}$. Multiplying F_4 by G_4 and discarding powers x^k with $k > 4$ yields $H_4(x) = x + x^2 + \dfrac{x^3}{3}$. Therefore, $h'(0) = 1$, $h''(0) = 2$, $h'''(0) = 2$, and $h^{(4)}(0) = 0$.

(ii) Let $f(x) = \cos\pi x/2$, $g(x) = \log x$. Then $F_4(x) = -\dfrac{\pi}{2}(x-1) + \dfrac{\pi^3}{48}(x-1)^3$ and $G_4(x) = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dfrac{(x-1)^4}{4}$. Multiplying F_4 by G_4 and discarding powers $(x-1)^k$ with $k > 4$ yields $H_4(x) = -\dfrac{\pi}{2}(x-1)^2 + \dfrac{\pi}{4}(x-1)^3 + \left(\dfrac{\pi^3}{48} - \dfrac{\pi}{6}\right)(x-1)^4$. Therefore, $h'(1) = 0$, $h''(1) = -\pi$, $h'''(1) = \dfrac{3\pi}{2}$, and $h^{(4)}(1) = -\pi + \dfrac{\pi^3}{2}$.

(iii) Let $f(x) = x^2$, $g(x) = \cos x$. Then $F_4(x) = \left(\dfrac{\pi}{2} + \left(x - \dfrac{\pi}{2}\right)\right)^2 = \dfrac{\pi^2}{4} + \pi\left(x-\dfrac{\pi}{2}\right) + \left(x-\dfrac{\pi}{2}\right)^2$ and $G_4(x) = -\left(x-\dfrac{\pi}{2}\right) + \dfrac{1}{6}\left(x-\dfrac{\pi}{2}\right)^3$. Multiplying F_4 by G_4 and discarding powers $\left(x-\dfrac{\pi}{2}\right)^k$ with $k > 4$ yields $H_4(x) = -\dfrac{\pi^2}{4}\left(x-\dfrac{\pi}{2}\right) - \pi\left(x-\dfrac{\pi}{2}\right)^2 + \left(\dfrac{\pi^2}{24} - 1\right)\left(x-\dfrac{\pi}{2}\right)^3 + \dfrac{\pi}{6}\left(x-\dfrac{\pi}{2}\right)^4$. Therefore, $h'(\pi/2) = -\dfrac{\pi^2}{4}$, $h''(\pi/2) = -2\pi$, $h'''(\pi/2) = -6 + \dfrac{\pi^2}{4}$, and $h^{(4)}(\pi/2) = 4\pi$.

(iv) Let $f(x) = (1+x)^{-1}$, $g(x) = e^{-x}$. Then $F_4(x) = 1 - x + x^2 - x^3 + x^4$ and $G_4(x) = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dfrac{x^4}{24}$. Multiplying F_4 by G_4 and discarding powers x^k with $k > 4$ yields $H_4(x) = 1 - 2x + \dfrac{5}{2}x^2 - \dfrac{8}{3}x^3 + \dfrac{65}{24}x^4$. Therefore, $h'(0) = -2$, $h''(0) = 5$, $h'''(0) = -16$, and $h^{(4)}(0) = 65$.

2.5.11. **(a)** By Lemma 2, $f(y) = F_n(y) + E(y)(y-y_0)^n$, where $\lim_{y\to y_0} E(y) = 0$. Hence,

$$\text{(A)}\quad \lim_{x\to x_0}\frac{f(g(x)) - F_n(g(x))}{(x-x_0)^n} = \lim_{x\to x_0} E_1(g(x)) \lim_{x\to x_0}\left(\frac{g(x)-g(x_0)}{x-x_0}\right)^n = 0\cdot(g'(x_0))^n = 0.$$

Since F_n' is bounded in some neighborhood of $g(x_0)$ and $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} G_n(x) =$

$g(x_0)$, Theorem 2.3.14 implies that there is a constant M such $|F_n(g(x)) - F_n(G_n(x))| \le M|g(x) - G_n(x)|$ for x in some neighborhood of x_0. Therefore, $\left|\dfrac{F_n(g(x)) - F_n(G_n(x))}{(x-x_0)^n}\right| \le M\left|\dfrac{g(x) - G_n(x))}{(x-x_0)^n}\right|$; hence (B) $\lim_{x\to x_0} \dfrac{F_n(g(x)) - F_n(G_n(x))}{(x-x_0)^n} = 0$, since $\lim_{x\to x_0} \dfrac{g(x) - G_n(x)}{(x-x_0)^n} = 0$ (Lemma 2.5.2). (A) and (B) imply that $\lim_{x\to x_0} \dfrac{f\,(g(x)) - F_n\,(G_n(x))}{(x-x_0)^n} = 0$, and the conclusion follows from Exercise 2.5.8**(b)**.

(b) $F_4(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ and $G_4(x)x - \dfrac{x^3}{6}$. Forming $F_4(G_4(x))$ and discarding powers x^k with $k > 4$ yields $H_4(x) = 1 - \dfrac{x^2}{4} + \dfrac{5x^2}{24}$. Therefore, $h'(0) = 0$, $h''(0) = -1$, $h'''(0) = 0$, and $h^{(4)}(0) = 5$.

2.5.12. **(a)** With $f(y) = 1/y$ and $y_0 = 1$, $F_n(y) = \sum_{r=0}^n (-1)^r(y-1)^r$. Apply Exercise 2.5.11**(a)**.

(b) **(i)** $g(x) = \sin x$, so $G_4(x) = 1 - \dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right) + \dfrac{1}{24}\left(x - \dfrac{\pi}{2}\right)^4$. Forming $\sum_{r=1}^4 [1 - G_4(x)]^r$ and discarding powers $\left(x - \dfrac{\pi}{2}\right)^k$ with $k > 4$ yields $H_4(x) = 1 + \dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right)^2 + \dfrac{5}{24}\left(x - \dfrac{\pi}{2}\right)^4$, so $h'(\pi/2) = 0$, $h''(\pi/2) = 1$, $h'''(\pi/2) = 0$, and $h^{(4)}(\pi/2) = 5$.

(ii) $g(x) = 1 + x + x^2$, so $G_4(x) = 1 + x + x^2$. Forming $\sum_{r=1}^4 [1 - G_4(x)]^r$ and discarding powers x^k with $k > 4$ yields $H_4(x) = 1 - x + x^3 - x^4$ so $h'(0) = -1$, $h''(0) = 0$, $h'''(0) = 6$, and $h^{(4)}(0) = -24$.

(iii) We first consider $h_0(x) = (\sec x)/\sqrt{2}$, for which $g_0(x) = \sqrt{2}\cos x$, which satisfies the normalization condition $g_0(\pi/2) = 1$. The fourth Talylor polynomial of g_0 about $\pi/4$ is $G_4(x) = 1 - \left(x - \dfrac{\pi}{4}\right) - \dfrac{1}{2}\left(x - \dfrac{\pi}{4}\right)^2 + \dfrac{1}{6}\left(x - \dfrac{\pi}{4}\right)^3 + \dfrac{1}{24}\left(x - \dfrac{\pi}{4}\right)^4$. Forming $\sum_{r=1}^4 [1 - G_4(x)]^r$ and discarding powers $\left(x - \dfrac{\pi}{4}\right)^k$ with $k > 4$ yields the fourth Taylor polynomial $H_4(x) = 1 + \left(x - \dfrac{\pi}{4}\right) + \dfrac{3}{2}\left(x - \dfrac{\pi}{4}\right)^2 + \dfrac{11}{6}\left(x - \dfrac{\pi}{4}\right)^3 + \dfrac{19}{8}\left(x - \dfrac{\pi}{4}\right)$ for $h/\sqrt{2}$, so $h'(\pi/4) = \sqrt{2}$, $h''(\pi/4) = 3\sqrt{2}$, $h'''(\pi/4) = 11\sqrt{2}$, and $h^{(4)}(\pi/4) = 57\sqrt{2}$.

(iv) $g(x) = 1 + \log(1+x)$, so $G_4(x) = 1 + x - \dfrac{x^2}{4} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$. Forming $\sum_{r=1}^4 [1 - G_4(x)]^r$ and discarding powers ^{k} with $k > 4$ yields $H_4(x) = 1 - x + \dfrac{3}{2}x^2 - \dfrac{7}{3}x^3 + \dfrac{11}{3}x^4$, so $h'(0) = -1$, $h''(0) = 3$, $h'''(0) = -14$, and $h^{(4)}(0) = 88$.

(c) Since $hg = 1$, which is its own Taylor polynomial for every n, Exercise 2.5.10 implies that $H_nG_n = 1+$ powers of $(x - x_0)$ higher than n. However, $H_n(x)G_n(x) = 1 +$

$\sum_{k=1}^{n} c_k(x-x_0)^k +$ powers of $x - x_0$ higher than n, with $c_k = \sum_{r=0}^{k} a_r b_{k-r}$, $1 \le k \le n$. Hence $\sum_{r=0}^{k} a_r b_{k-r} = 0$, $1 \le k \le n$. This implies the result.

2.5.13. **(a)** $f'(x) = (3x^4 + 2x)e^{x^3}$, so $f'(0) = 0$; $f''(x) = (9x^6 + 18x^3 + 2)e^{x^3}$, so $f''(0) > 0$; hence 0 is a local minimum of f.

(b) $f'(x) = (3x^5 + 3x^2)e^{x^3}$, so $f'(0) = 0$; $f''(x) = (9x^7 + 24x^4 + 6x)e^{x^3}$, so $f''(0) = 0$; $f'''(x) = (27x^9 + 135x^6 + 114x^3 + 6)e^{x^3}$, so $f'''(0) = 6$; hence 0 is not a local extreme point of f.

(c) $f'(x) = -\dfrac{x(x^3+3x-2)}{(x^3+1)^2}$, so $f'(0) = 0$; $f''(x) = \dfrac{2(x^6+6x^4-7x^3-3x+1)}{(x^3+1)^3}$, so $f''(0) = 2$; hence 0 is a local minimum of f.

(d) $f'(x) = \dfrac{x(x^3+3x-2)}{(x^2+1)^2}$, so $f'(0) = 0$; $f''(x) = \dfrac{-2(x^3-3x^2-3x+1)}{(x^2+1)^3}$, so $f''(0) = -2$; hence 0 is a local maximum of f.

(e) $f'(x) = (3x^2\sin^2 x + 2x)\cos x + 2x\sin^3 x - x^2\sin x$, so $f'(0) = 0$; $f''(x) = 6x^2\sin x\cos^2 x + (12x\sin^2 x - x^2 + 2)\cos x + (2-3x^2)\sin^3 x - 4x\sin x$, so $f''(0) = 2$; hence 0 is a local minimum of f.

(f) $f'(x) = e^{x^2}(\cos x + 2x\sin x)$, so $f'(0) = 1$; hence 0 is neither a local maximum nor a local mininum.

(g) $f'(x) = e^x\left(2x\cos(x^2) + \sin(x^2)\right)$, so $f'(0) = 0$;

$$f''(x) = e^x\left((4x+2)\cos(x^2) + (1-4x^2)\sin(x^2)\right),$$

so $f''(0) = 2$; hence 0 is a local minimum of f.

(h) $f'(x) = e^{x^2}(2x\cos x - \sin x)$, so $f'(0) = 0$; $f''(x) = e^{x^2}((4x^2+1)\cos x - 4x\sin x)$, so $f''(0) = 1$; hence 0 is a local minimum of f.

2.5.14. If $f(x) = e^{-1/x^2}$ if $x \ne 0$ and $f(0) = 0$ (minimum value) then f is infinitely differentiable for all x and $f^{(n)}(0) = 0$ for all n. (See Exercise 2.4.40 and its solution.)

2.5.15. Since $f'(x) = x^2 + bx + c$ and $f''(x) = 2x + b$, there are three cases: (i) If $b^2 < 4c$, then f' has no zeros, so f has no relative extrema. (ii) If $b^2 > 4c$, the zeros of f' are $x_1 = \dfrac{-b+\sqrt{b^2-4c}}{2}$ and $x_2 = \dfrac{-b-\sqrt{b^2-4c}}{2}$. Since $f''(x_1) = \sqrt{b^2-4c} > 0$ and $f''(x_2) = -\sqrt{b^2-4c} < 0$, $f(x_1)$ is a relative minimum and $f(x_2)$ is a relative maximum of f. (iii) If $b^2 = 4c$, then f' has the repeated zero $r_1 = -b/2$. Therefore, $f'(x) = (x-r_1)^2$, so $f'(r_1) = f''(r_1) = 0$ while $f'''(r_1) = 2$, so f has no relative extreme points.

2.5.16. **(a)** $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$. Therefore, $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$, so Theorem 2.5.4 with $x_0 = 0$ and $n = 2$ implies that $\sin x = x - \dfrac{\cos c}{6}x^3$ for some c between 0 and x. Since $|\cos c| \le 1$, this implies that

$|\sin x - x| < \dfrac{1}{6}\left(\dfrac{\pi}{20}\right)^3$ if $|x| < \dfrac{\pi}{20}$.

(b) $f(x) = \sqrt{1+x}$, $f'(x) = \dfrac{1}{2\sqrt{1+x}}$, $f''(x) = -\dfrac{1}{4(x+1)^{3/2}}$. Therefore, $f(0) = 0$ and $f'(0) = \dfrac{1}{2}$, so Theorem 2.5.4 with $x_0 = 0$ and $n = 1$ implies that $\sqrt{1+x} = 1 + \dfrac{x}{2} + \dfrac{x^2}{8(c+1)^{3/2}}$ for some c between 0 and x. Therefore, $\left|\sqrt{1+x} - 1 - \dfrac{x}{2}\right| < \dfrac{1}{8^3}$ if $|x| < \dfrac{1}{8}$.

(c) $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$ Therefore, $f(\pi/4) = \dfrac{1}{\sqrt{2}}$ and $f'(\pi/2) = \dfrac{1}{\sqrt{2}}$, so Theorem 2.5.4 with $x_0 = \dfrac{\pi}{2}$ and $n = 1$ implies that $\cos x = \dfrac{1}{\sqrt{2}}\left[1 - \left(x - \dfrac{\pi}{4}\right)\right] - \dfrac{\cos c}{2}\left(x - \dfrac{\pi}{4}\right)^2$ for some c between $\dfrac{\pi}{4}$ and x. Since $|\cos c| \le \dfrac{1}{\sqrt{2}}$ if $\dfrac{\pi}{4} < x < \dfrac{5\pi}{16}$, this implies that $\left|\cos x - \dfrac{1}{\sqrt{2}}\left[1 - \left(x - \dfrac{\pi}{4}\right)\right]\right| < \dfrac{\pi^2}{512\sqrt{2}}$ if $\dfrac{\pi}{4} < x < \dfrac{5\pi}{16}$.

(d) $f(x) = \log x$, $f'(x) = \dfrac{1}{x}$, $f''(x) = -\dfrac{1}{x^2}$, $f'''(x) = \dfrac{2}{x^3}$, $f^{(4)}(x) = -\dfrac{6}{x^4}$. Therefore, $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, and $f'''(1) = 2$ so Theorem 2.5.4 with $x_0 = 1$ and $n = 3$ implies that $\log x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} + R_3(x)$, with $R_3(x) = -\dfrac{1}{4}\left(\dfrac{x-1}{c}\right)^4$ for some c between 1 and x. Therefore, $|R_3(x)| < \dfrac{1}{4(64)^4}\left(\dfrac{1}{1-1/64}\right)^4 = \dfrac{1}{4(63)^4}$ if $|x-1| < \dfrac{1}{64}$.

2.5.17. Since $T_{n+1}(x) = T_n(x) + \dfrac{x^{n+1}}{(n+1)!}$, $T_n(x) < T_{n+1}(x)$ if $x > 0$. If $n \ge 0$ Theorem 2.5.4, implies that $e^x = T_n(x) + e^{c_m}x^{n+1}/(n+1)!$ where $0 < c_n < x$. Since $1 < e^{c_n} < e^x$ if $0 < c_n < x$, this implies that if $x > 0$ then $e^x > T_{n+1}(x)$ and $e^x < T_n(x) + e^x\dfrac{x^{n+1}}{(n+1)!}$. The last inequality implies that $e^x\left[1 - \dfrac{x^{n+1}}{(n+1)!}\right]$, so $e^x < \left[1 - \dfrac{x^{n+1}}{(n+1)!}\right]^{-1} T_n(x)$ if $0 < x < [(n+1)!]^{1/(n+1)}$.

2.5.18. **(a)** To verify P_1:

$$\begin{aligned}\Delta[c_1 f_1(x) + \cdots + c_k f_k(x)] &= c_1 f_1(x+h) + \cdots + c_k f_k(x+h)\\ &\quad - c_1 f_1(x) - \cdots - c_k f_k(x)\\ &= c_1 \Delta f_1(x) + \cdots + c_k \Delta f_k(x).\end{aligned}$$

Now suppose that P_n is true for $n \geq 1$. Then

$$\begin{aligned}\Delta^{n+1}[c_1 f_1(x) + \cdots + c_k f_k(x)] &= \Delta\left(\Delta^n[c_1 f_1(x) + \cdots + c_k f_k(x)]\right)\\ &= \Delta\left[c_1\Delta^n f_1(x) + \cdots + c_k\Delta^n f_k(x)\right]\\ &= c_1\Delta(\Delta^n f_1(x)) + \cdots + c_k\Delta\left(\Delta^n f_k(x)\right)\\ &= c_1\Delta^{n+1} f_1(x) + \cdots + c_k\Delta^{n+1} f_k(x),\end{aligned}$$

where the second equality follows from P_n, the third from P_1. This verifies P_{n+1}.
(b) P_1 is just the definition of $\Delta f(x)$. Now suppose that $n \geq 1$ and P_n is true. Then

$$\Delta^{n+1} f(x) = \Delta(\Delta^n f(x)) = \sum_{m=0}^{n} (-1)^{n-m}\binom{n}{m}\Delta f(x+mh)$$

(by P_n and **(a)**). Therefore,

$$\begin{aligned}\Delta^{n+1} f(x) &= \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}[f\,(x+(m+1)h) - f(m+kh)]\\ &= \sum_{m=1}^{n+1}(-1)^{n+1-m}\binom{n}{m} - 1 f(x+mh) - \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m} f(x+km)\\ &= (-1)^{n+1} f(x) + \sum_{m=1}^{n}(-1)^{n+1-m}\left[\binom{n}{m} - 1 + \binom{n}{m}\right] f(x+mh)\\ &+ f\,(x+(n+1)h)\\ &= \sum_{m=0}^{n+1}(-1)^{n+1-m}\binom{n+1}{m} f(x+mh),\end{aligned}$$

which verifies P_{n+1}.
2.5.19. Since $\Delta^0(x-x_0)^0 = \Delta^0 1 = 1$, P_0 is true. Since $\Delta 1 = 1-1 = 0$ and $\Delta(x-x_0) = (x+h-x_0)-(x-x_0) = h$, P_1 is true. Now suppose that $n \geq 1$ and P_n is true. If $0 \leq m < n$, then $\Delta^{n+1}(x-x_0)^m = \Delta\left(\Delta^n(x-x_0)^m\right) = \Delta 0$ (by P_n)$= 0\Delta 1 = 0$, by P_1. Also, $\Delta^{n+1}(x-x_0)^n = \Delta\left(\Delta^n(x-x_0)^n\right) = h^n n!$ (by P_n) $= h^n n!\Delta 1 = 0$, by P_1. To complete the induction we must show that (A) $\Delta^{n+1}(x-x_0)^{n+1} = (n+1)!h^{n+1}$. From the binomial theorem (Exercise 1.2.19), $(x+h-x_0)^{n+1} = \sum_{m=0}^{n+1}\binom{n+1}{m}h^{n+1-m}(x-x_0)^m$, which implies that $\Delta(x-x_0)^{n+1} = (n+1)h(x-x_0)^n + \sum_{m=0}^{n-1}\binom{n+1}{m}h^{n+1-m}(x-x_0)^m$. Therefore, $\Delta^{n+1}(x-x_0)^{n+1} = (n+1)h\Delta^n(x-x_0)^n + \sum_{m=0}^{n-1}\binom{n+1}{m}h^{n+1-m}\Delta^n(x-x_0)^m$. Now P_n and Exercise 2.5.18 imply that $\Delta^{n+1}(x-x_0)^{n+1} = (n+1)h(n!h^n) = (n+1)!h^{n+1}$. This completes the induction.

2.5.20. **(a)** By Theorem 2.5.4,

$$f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(c_1)\frac{h^3}{6},$$
$$f(x_0-h) = f(x_0) - f'(x_0)h + f''(x_0)\frac{h^2}{2} - f'''(c_2)\frac{h^3}{6},$$

where $x_0 < c_1 < x_0 + h$ and $x_0 - h < c_2 < x_0$. Therefore,

$$\frac{\Delta^2 f(x_0-h)}{h^2} = f''(x_0) + \frac{h}{6}[f''(c_1) - f''(c_2)],$$

so

$$\left|\frac{\Delta^2 f(x_0-h)}{h^2} - f''(x_0)\right| \le \frac{M_3 h}{3},$$

where $M_3 = \sup_{|x_0-c|<h} |f'''(c)|$.
(b) By Theorem 2.5.4,

$$f(x_0+h) = \sum_{r=0}^{3} f^{(r)}(x_0)\frac{h^r}{r!} + f^{(4)}(c_1)h^4 24,$$

where $x_0 < c_1 < x_0 + h$, and

$$f(x_0-h) = \sum_{r=0}^{3} f^{(r)}(x_0)\frac{h^r}{r!} + f^{(4)}(c_2)\frac{h^4}{24},$$

where $x_0 - h < c_2 < x_0$. Therefore,

$$\frac{\Delta^2 f(x_0-h)}{h^2} = f''(x_0) + \left[f^{(4)}(c_1) + f^{(4)}(c_2)\right]\frac{h^2}{24},$$

so

$$\left|\frac{\Delta^2 f(x_0-h)}{h^2} - f''(x_0)\right| \le \frac{M_4 h^2}{12},$$

where

$$M_4 = \sup_{|x_0-c|<h} |f''(c)|.$$

2.5.21. From Theorem 2.5.4,

$$f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(c_0)\frac{h^3}{6}, \qquad \text{(A)}$$

where $x_0 < c_0 < x_0 + h$. Solving this for $f'(x_0)$ yields

$$f'(x_0) = \frac{\Delta f(x_0)}{h} - f''(x_0)\frac{h}{2} - f'''(c_0)\frac{h^2}{6}. \qquad \text{(B)}$$

Now we must express $f''(x_0)$ in terms of $\Delta^2 f(x_0)$ and values of f'''. To this end we apply Theorem 2.5.4 again to write

$$f(x_0+2h) = f(x_0) + 2f'(x_0)h + 2f''(x_0)h^2 + \frac{4}{3}f'''(c_1)h^3$$

where $x_0 < c_1 < x_0 + 2h$. Subtracting twice the equation (A) from this yields

$$f(x_0+2h) - 2f(x_0+h) = -f(x_0) + f''(x_0)h^2 + \frac{4}{3}f'''(c_1)h^3 - \frac{1}{3}f'''(c_0)h^3.$$

Solving this for $f''(x_0)$ yields

$$f''(x_0) = \frac{\Delta^2 f(x_0)}{h^2} - \frac{4}{3}f'''(c_1)h + \frac{1}{3}f'''(c_0)h. \qquad \text{(C)}$$

Substituting (C) into (B) yields

$$f'(x_0) = \frac{\Delta f(x_0)}{h} - \frac{1}{2h}\Delta^2 f(x_0) + \left(\frac{2}{3}f'''(c_1) - \frac{1}{3}f'''(c_0)\right)h^2;$$

hence $k = -h/2$.

2.5.22. If $m \geq 0$, then Theorem 2.5.4 implies that

$$f(x_0+mh) = \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(mh)^r + \frac{f^{(n+1)}(c_m)}{(n+1)}(mh)^{n+1},$$

where $x_0 < c_m < x_0 + mh$. In particular, this is true if $0 \leq m \leq n$. Therefore,

(A) $\displaystyle \Delta^n f(x_0) = \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}h^r \Delta^n m^r + \frac{1}{(n+1)!}\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(mh)^{n+1}f^{(n+1)}(c_m).$

From Exercise 2.5.19, $\Delta^n m^r = \begin{cases} 0 & \text{if } 0 \leq m \leq n, \\ n!h^n & \text{if } m = n. \end{cases}$, so $\displaystyle\sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}h^r \Delta^n m^r =$ $f^{(n)}(x_0)h^n$. Now (A) implies the stated inequality, with $\displaystyle A_n = \frac{1}{(n+1)!}\sum_{m=0}^{n}\binom{n}{m}m^{n+1}$.

2.5.23. If $x = x_i$ for some i, the relation holds for any c. If not, then $g(y)$ has at least $n+2$ distinct zeros in (a,b). Repeated applications of Rolle's theorem imply that $g^{(r)}$ has at least $n+2-r$ zeros in (a,b). In particular, $g^{(n+1)}(c) = 0$ for some c in (a,b). Since $p^{(n+1)} = 0$, $g^{(n+1)}(y) = f^{(n+1)}(y) - K$. Setting $y = c$ yields the result.

2.5.24. Take $a = x_0$ and $b = x$ in Theorem 2.5.5.

CHAPTER 3
Integral Calculus of Functions of One Variable

3.1 DEFINITION OF THE INTEGRAL

3.1.1. Suppose that L_1 and L_2 both have the properties required of L in Definition 3.1.1. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that if σ is Riemann sum of f over any partition P of $[a,b]$ with $\|P\| < \delta$, then $|\sigma - L_1| < \epsilon$ and $|\sigma - L_2| < \epsilon$. Therefore, $|L_2 - L_1| = |L_1 - \sigma + \sigma - L_2| \le |\sigma - L_2| + |\sigma - L_1| < 2\epsilon$. Since ϵ can be chosen arbitrarily small, it follows that $L_1 = L_2$.

3.1.2. **(a)** Suppose that $\displaystyle\int_a^b f(x)\,dx$ exists and let $\epsilon > 0$. Choose $\delta > 0$ so that $|\sigma - \int_a^b f(x)\,dx| < \epsilon/2$ if σ is any Riemann sum of f over a partition of $[a,b]$ with $\|P\| < \delta$. Now suppose that σ_1 and σ_2 are Riemann sums of f over partitions P_1 and P_2 with norms less than δ. Then

$$\begin{aligned}|\sigma_1 - \sigma_2| &= \left|\sigma_1 - \int_a^b f(x)\,dx + \int_a^b f(x)\,dx - \sigma_2\right| \\ &\le \left|\sigma_1 - \int_a^b f(x)\,dx\right| + \left|\int_a^b f(x)\,dx - \sigma_2\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

(b) Part (a) implies that if $\epsilon > 0$, there is a $\delta > 0$ such that if σ and σ' are Riemann sums of f over a partition P with $\|P\| < \delta$, then $|\sigma - \sigma'| < \epsilon$. Just choose $\epsilon < M$ to show that f is not integrable over $[a,b]$.

3.1.3. For a given $\epsilon > 0$ we can choose δ so that $\left|\sigma - \int_a^b f(x)\,dx\right| < \epsilon$ if $\|P\| < \delta$ and σ is any Riemann sum of f over P (Definition 3.1.1). Then choose P so that $\|P\| < \delta$ and there is a Riemann sum σ of f over P such that $|\sigma - A| < \epsilon$. Then $\left|A - \int_a^b f(x)\,dx\right| \le$

$|A-\sigma| + |\sigma - \int_a^b f(x)\,dx| < 2\epsilon$. Since ϵ can be chosen arbitrarily small, this implies that $A = \int_a^b f(x)\,dx$

3.1.4. By the mean value theorem $(x_j^3 - x_{j-1}^3)/3 = d_j^2(x_j - x_{j-1})$ for some d_j in (x_{j-1}, x_j). Then

$$\sum_{j=1}^n d_j^2(x_j - x_{j-1}) = \sum_{j=1}^n \frac{x_j^3 - x_{j-1}^3}{3} = \frac{b^3 - a^3}{3}.$$

Now let $\sigma = \displaystyle\sum_{j=1}^n c_j^2(x_j - x_{j-1})$ be an arbitrary Riemann sum of $f(x) = x^2$ over P. Since $x_{j-1} < c_j < x_j$, $|c_j^2 - d_j^2| = |c_j - d_j|\,|c_j + d_j| \le 2\|P\|\max(|a|,|b|)$, so $|\sigma - (b^3 - a^3)/3| \le 2\|P\|(b-a)\max(|a|,|b|)$. Since $\|P\|$ can can be chosen arbitrarily small, this implies the conclusion.

3.1.5. Let $P = \{x_0, x_2, \dots, x_n\}$ be an arbitrary partition of $[a,b]$. By the mean value theorem, $(x_j^{m+1} - x_{j-1}^{m+1})/(m+1) = d_j^m(x_j - x_{j-1})$ for some d_j in (x_{j-1}, x_j). Then

$$\sum_{j=1}^n d_j^m(x_j - x_{j-1}) = \sum_{j=1}^n \frac{x_j^{m+1} - x_{j-1}^{m+1}}{3} = \frac{b^{m+1} - a^{m+1}}{3}.$$

Now let $\sigma = \displaystyle\sum_{j=1}^n c_j^m(x_j - x_{j-1})$ be an arbitrary Riemann sum of $f(x) = x^m$ over P. Then

$$|c_j^m - d_j^m| = |c_j - d_j| \left|\sum_{r=1}^{m-1} c_j^r d_j^{m-r-1}\right| \le \|P\|(m-1)A^{m-1},$$

where $A = \max\{|a|,|b|\}$. Therefore,

$$\left|\sigma - \frac{b^{m+1} - a^{m+1})}{m+1}\right| \le \|P\|(m-1)A^{m-1}(b-a).$$

Since $\|P\|$ can be chosen arbitrarily small, this implies the conclusion.

3.1.6. Let $\sigma = \sum_{j=1}^n f(c_j)(x_j - x_{j-1})$ be a Riemann sum for f over $[a,b]$. Define $x_j' = -x_{n-j}$, $0 \le j \le n$, and $c_j' = -c_{n-j+1}$, $1 \le j \le n$. Then we can rewrite σ as $\sigma = \sum_{j=1}^n f(-c_j')(x_j' - x_{j-1}')$; that is, every Riemann sum of $f(x)$ over a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ is a Riemann sum of $f(-x)$ over the partition $P' = \{x_0', x_1', \dots, x_n'\}$ of $[-b,-a]$. The converse is also true, by the same argument. Therefore, Definition 3.1.1 implies the conclusion.

3.1.7. If $P = \{x_0, x_1, \dots, x_n\}$, then $s(P) = \displaystyle\sum_{j=1}^n m_j(x_j - x_{j-1})$, where $m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$.

An arbitrary Reimann sum of f over P is of the form $\sigma = \displaystyle\sum_{j=1}^n f(c_j)(x_j - x_{j-1})$ where $x_{j-1} \le c_j \le x_j$. Since $f(c_j) \ge m_j$, $\sigma \ge s(P)$.

Now let $\epsilon > 0$ and choose $\overline{c}_j$ in $[x_{j-1}, x_j]$ so that $f(\overline{c}_j) < m_j + \dfrac{\epsilon}{n(x_j - x_{j-1})}$, $1 \le j \le n$. The Riemann sum produced in this way is $\overline{\sigma} = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) > \sum_{j=1}^{n} \left[m_j + \dfrac{\epsilon}{n(x_j - x_{j-1})}) \right] (x_j - x_{j-1}) = s(P) + \epsilon$. Now Theorem 1.1.8 implies that $s(P)$ is the infimum of the set of Riemann sums of f over P.

3.1.8. **(a)** From Theorem 2.2.9, for $1 \le j \le n$, there are points a_j and b_j in $[x_{j-1}, x_j]$ such that $f(a_j) = M_j$ and $f(b_j) = m_j$. Therefore, $S(P) = \sum_{j=1}^{n} f(a_j)(x_j - x_{j-1})$ and $s(P) = \sum_{j=1}^{n} f(b_j)(x_j - x_{j-1})$.

3.1.9. **(a)** Every lower sum of $f(x)$ is a Riemann sum for $g(x) = -x$, so $\underline{\int_0^1} f(x)\,dx = -\int_0^1 x\,dx = -\dfrac{1}{2}$. Every upper sum of $f(x)$ is a Riemann for $h(x) = x$, so $\overline{\int_0^1} f(x)\,dx = \int_0^1 x\,dx = \dfrac{1}{2}$.

(b) Every lower sum of $f(x)$ is a Riemann sum for $g(x) = x$, so $\underline{\int_0^1} f(x)\,dx = \int_0^1 x\,dx = \dfrac{1}{2}$. Every upper sum of $f(x)$ is a Riemann sum for $h(x) = 1$, so $\overline{\int_0^1} f(x)\,dx = \int_0^1 dx = 1$.

3.1.10. Let $x_j = a + j(b-a)/n$ and

$$\begin{aligned}
\sigma &= \sum_{j=1}^{n} e^{x_{j-1}}(x_j - x_{j-1}) = e^a \frac{(b-a)}{n} \sum_{j=1}^{n} \exp\left[(j-1)(b-a)/n\right] \\
&= e^a \frac{(b-a)}{n} \frac{1 - e^{(b-a)}}{1 - e^{(b-a)/n}} = (e^b - e^a) \frac{(b-a)/n}{e^{(b-a)/n} - 1}.
\end{aligned}$$

Since $\lim_{x \to 0} x/(e^x - 1) = 1$, Exercise 3.1.3 implies that $\int_a^b e^x\,dx = e^b - e^a$.

3.1.11. Let $x_j = jb/n$ and consider the Riemann sum

$$\begin{aligned}
\sigma &= \sum_{j=1}^{n} \sin x_{j-1}(x_j - x_{j-1}) = \frac{b}{n} \sum_{j=2}^{n} \sin(j-1)\frac{b}{n} \\
&= \frac{b/n}{2 \sin b/n} \sum_{j=2}^{n} \left[\cos(j-2)\frac{b}{n} - \cos\frac{jb}{n} \right] \\
&= \frac{b/n}{2 \sin b/n} \left[1 + \cos\frac{b}{n} - \cos(n-1)\frac{b}{n} - \cos b \right],
\end{aligned}$$

which approaches $1 - \cos b$ as $n \to \infty$. Now use Exercise 3.1.3.

3.1.12. Let $x_j = jb/n$ and and consider the Riemann sum

$$\begin{aligned}\sigma &= \sum_{j=1}^{n} \cos x_j (x_j - x_{j-1}) = \frac{b}{n} \sum_{j=1}^{n} \cos \frac{jb}{n} \\ &= \frac{b/n}{2 \sin b/n} \sum_{j=2}^{n} \left[\sin(j+1)\frac{b}{n} - \sin(j-1)\frac{b}{n} \right] \\ &= \frac{b/n}{2 \sin b/n} \left[-\sin \frac{b}{n} + \sin b + \sin(n-1)\frac{b}{n} \right],\end{aligned}$$

which approaches $\sin b$ as $n \to \infty$. Now use Exercise 3.1.3.

3.1.13. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Then every Riemann - Stieltjes sum of f with respect to g over P,

$$\sum_{j=1}^{n} f(c_j) \left[g(x_j) - g(x_{j-1}) \right] = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}),$$

is a Riemann sum of f over P, and conversely. This implies the conclusion.

3.1.14. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ such that $\|P\| < \min(d-a, b-d)$. Let $c_1, c_2, \dots, c_n$ be the intermediate points occurring in a Riemann - Stieltjes sum of f with respect to g over P; that is, $x_{j-1} \le c_j \le x_j$, $1 \le j \le n$. If $d \notin [x_{j-1}, x_j]$, then $g(x_j) - g(x_{j-1}) = 0$. Therefore, if $x_{i-1} < d < x_i$ for some i in $\{1, 2, \dots, n\}$, then $\sigma = f(c_1)[g_1 - g(a)] + f(c_i)(g_2 - g_1) + f(c_n)[g(b) - g_2]$. On the other hand, if $c = x_i$ for some i in $\{1, 2, \dots, n\}$, then $\sigma = f(c_1)[g_1 - g(a)] + f(c_{i-1})[g(d) - g_1)] + f(c_i)[g_2 - g(d)] + f(c_n)[g(b) - g_2]$. From the continuity assumptions on f, in either case $\sigma \to f(a)[g_1 - g(a)] + f(d)(g_2 - g_1) + f(b)[g(b) - g_2] = \int_a^b f(x)\, dg(x)$ as $\|P\| \to 0$.

3.1.15. See the proof of Exercise 3.1.14.

3.1.16. **(b)** If g is increasing and f is unbounded on $[a, b]$, then $\displaystyle\int_a^b f(x)\, dg(x)$ does not exist. (See the proof of Theorem 3.1.2.)

3.1.17. Same as Definition 3.1.3, except that $x_j - x_{j-1}$ is replaced by $g(x_j) - g(x_{j-1})$.

3.2 EXISTENCE OF THE INTEGRAL

3.2.1. First suppose $r = 1$, so P' is obtained by adding one point c to the partition $P = \{x_0, x_1, \dots, x_n\}$; then $x_{i-1} < c < x_i$ for some i in $\{1, 2, \dots, n\}$. If $j \ne i$, then the product $r_j(x_j - x_{j-1})$ appears in both $s(P')$ and $s(P)$, and cancels out of the difference $s(P') - s(P)$. Therefore, if

$$m_{i1} = \inf_{x_{i-1} \le x \le c} f(x) \quad \text{and} \quad m_{i2} = \inf_{c \le x \le x_i} f(x),$$

then

$$\begin{aligned}s(P') - s(P) &= m_{i1}(c - x_{i-1}) - m_{i2}(x_i - c) - m_i(x_i - x_{i-1}) \\ &= (m_{i1} - m_i)(c - x_{i-1}) + (m_{i2} - m_i)(x_i - c).\end{aligned} \tag{A}$$

Since (1) implies that $0 \le r_{ir} - m_i \le 2M$, $r = 1, 2$, (A) implies that

$$0 \le s(P') - s(P) \le 2M(x_i - x_{i-1}) \le 2M\|P\|.$$

This proves (3) for $r = 1$.

Now suppose $r > 1$ and P' is obtained by adding points $c_1, c_2, \dots, c_r$ to P. Let $P^{(0)} = P$, and, for $j \ge 1$ let $P^{(j)}$ be the partition of $[a, b]$ obtained by adding c_j to $P^{(j-1)}$. Then the result just proved implies that

$$0 \le s(P^{(j)}) - s(P^{(j-1)}) \le 2M\|P^{(j-1)}\|, \quad 1 \le j \le r.$$

Adding these inequalities and taking account of cancellations that occur yields

$$0 \le s(P^{(r)}) - s(P^{(0)}) \le 2M(\|P^{(0)}\| + \|P^{(1)}\| + \cdots + \|P^{(r-1)}\|). \qquad \text{(B)}$$

Since $P^{(0)} = P$, $P^{(r)} = P'$ and $\|P^{(k)}\| \le \|P^{(k-1)}\|$ for $1 \le k \le r-1$, (B) implies that

$$0 \le s(P') - s(P) \le 2Mr\|P\|,$$

which is equivalent to (2).

3.2.2. Suppose that P is a partition of $[a, b]$ and σ is a Riemann sum of f over P. From the triangle inequality, (A) $\left|\underline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| \le \left|\underline{\int_a^b} f(x)\,dx - s(P)\right| + |s(P) - \sigma| + \left|\sigma - \int_a^b f(x)\,dx\right|$. Now suppose $\epsilon > 0$. From Definition 3.1.3, there is a partition P_0 of $[a, b]$ such that (B) $\underline{\int_a^b} f(x)\,dx \ge s(P_0) > \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{3}$. From Definition 3.1.1, there is a $\delta > 0$ such that (C) $\left|\sigma - \int_a^b f(x)\,dx\right| < \frac{\epsilon}{3}$ if $\|P\| < \delta$. Now suppose $\|P\| < \delta$ and P is a refinement of P_0. Since $s(P) \ge s(P_0)$ by Lemma 3.2.1, (B) implies that $\underline{\int_a^b} f(x)\,dx \ge s(P) > \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{3}$, so (D) $\left|s(P) - \underline{\int_a^b} f(x)\,dx\right| < \frac{\epsilon}{3}$ in addition to (C). Now (A), (C), and (D) imply that (E) $\left|\underline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \frac{2\epsilon}{3} + |s(P) - \sigma|$ for every Riemann sum σ of f over P. Since $s(P)$ is the infimum of these Riemann sums (Theorem (3.1.4)), we may choose σ so that $|s(P) - \sigma| < \frac{\epsilon}{3}$. Now (E) implies that $\left|\underline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \epsilon$. Since ϵ is an arbitrary positive number, it follows that $\underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$.

3.2.3. The first inequality follows immediately from Definition 3.1.3. To establish the second inequality, suppose $|f(x)| \le K$ if $a \le x \le b$. From Definition 3.1.3, there is a partition $P_0 = \{x_0, x_1, \dots, x_{r+1}\}$ of $[a,b]$ such that (A) $s(P_0) > \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2}$. If P is any partition of $[a,b]$, let P' be constructed from the partition points of P_0 and P. Then (B) $s(P') \ge s(P_0)$, by Lemma 3.2.1. Since P' is obtained by adding at most r points to P, Lemma 3.2.1 implies that (C) $s(P') \le s(P) + 2Kr\|P\|$ Now (A), (B), and (C) imply that $s(P) \ge s(P') - 2Kr\|P\| \ge s(P_0) - 2Kr\|P\| > \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2} - 2Kr\|P\|$ for every partition P. Therefore, $s(P) > \underline{\int_a^b} f(x)\,dx - \epsilon$ if $\|P\| < \delta = \frac{\epsilon}{4Kr}$.

3.2.4. Let $\int_a^b f(x)\,dx = L$. If $\epsilon > 0$, there is a $\delta > 0$ such that $L - \epsilon/3 < \sigma < L + \epsilon/3$ if σ is any Riemann sum of f over a partition P with $\|P\| < \delta$. Since $s(P)$ and S(P) are respectively the infimum and supremum of all Riemann sums of f over P (Theorem 3.1.4), $L-\epsilon/3 \le s(P) \le S(P) \le L+\epsilon/3$ if $\|P\| < \delta$. Therefore, $|S(P)-s(P)| < \epsilon$ if $\|P\| < \delta$.

3.2.5. (Quantities with subscripts f and g refer to f and g, respectively.) We first show that g is integrable on $[a,b]$. Let $\epsilon > 0$. From Theorem 3.2.7, there is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that (A) $S_f(P) - s_f(P) < \epsilon$. Let S be the set of integers in $\{1, \dots, n\}$ such that $[x_{j-1}, x_j]$ contains points from H. Since (A) remains valid if P is refined, we may assume that (B) $\sum_{j\in S}(x_{j-1} - x_j) < \epsilon$. Now suppose $|f(x)| \le M$ and $|g(x)| \le M$. Since $f(x) = g(x)$ for $x \in [x_{j-1}, x_j]$ if $j \notin S$, $\left|\big(S_g(P) - s_g(P)\big) - \big(S_f(P) - s_f(P)\big)\right| = \left|\sum_{j\in S}\big[(M_{g,j} - m_{g,j}) - (M_{f,p} - m_{f,p})\big]\right| \le 4M\sum_{j\in S}(x_{j-1} - x_j) < 4M\epsilon$, from (B). This and (A) imply that $S_g(P) - s_g(P) < S_f(P) - s_f(P) + 4M\epsilon < (4M+1)\epsilon$. Therefore, g is integrable over $[a,b]$, by Theorem 3.2.7.

To complete the proof we apply Exercise 3.1.3 to g, with $A = \int_a^b f(x)\,dx$. Let $\epsilon > 0$ and $\delta > 0$ be given. Let P be a partition such that: **(i)** $\|P\| < \delta$; **(ii)** $|\sigma_f - A| < \epsilon$ if σ_f is any Riemann sum of f of P; and **(iii)** (B) holds. Now let σ_g and σ_f be Riemann sums over P corresponding to the same choice of the intermediate points $c_0, c_1, \dots, c_n$. Then $|\sigma_g - \sigma_f| = \left|\sum_{j\in S}[g(c_j) - f(c_j)](x_j - x_{j-1})\right| \le 2M\sum_{j\in S}(x_{j-1} - x_j) < 2M\epsilon$. Therefore, $|\sigma_g - A| = |(\sigma_g - \sigma_f) + (\sigma_f - A)| \le |\sigma_g - A| = |\sigma_g - \sigma_f| + |\sigma_f - A| \le (2M+1)\epsilon$, which completes the proof (Exercise 3.1.3).

3.2.6. If P is an arbitrary partition of $[\alpha, \beta]$, let $s(P)$ and $S(P)$ be lower and upper sums of g over P. Let $\widehat{P}$ be the partition of $[\alpha, \beta]$ containing $v_0, v_1, \dots, v_L$ and the partition points of P. For $1 \le \ell \le L$ let P_ℓ be the partition of $[v_{\ell-1}, v_\ell]$ constructed from the partition points of of $\widehat{P}$ contained in that interval. Let $s(P_\ell)$ and $S(P_\ell)$ be the lower and upper sums of g over P_ℓ. Let $\epsilon > 0$.

(a) Let $Q_1 = \sum_{\ell=1}^{L} \underline{\int_{v_{\ell-1}}^{v_\ell}} g(u)\,du$. From Lemma 3.2.1, (A) $s(P) \le s(\widehat{P}) = \sum_{\ell=1}^{L} s_\ell(P)$. From Theorem 1.1.3, and Definition 3.1.3, $s_\ell(P) \le \underline{\int_{v_{\ell-1}}^{v_\ell}} g(u)\,du$. Therefore, (A) implies that $s(P) \le Q_1$. Moreover, again from Definition 3.1.3 and Theorem 1.1.3, there is a a partition P_ℓ of $[v_{\ell-1}, v_\ell]$ such that $s(P_\ell) > \underline{\int_{v_{\ell-1}}^{v_\ell}} g(u)\,du - \frac{\epsilon}{L}$; hence, if P is the partition of $[\alpha, \beta]$ constructed from the points in $P_1, P_2, \dots, P_L$, then $s(P) > Q_1 - \epsilon$. Hence, $Q_1 = \underline{\int_{\alpha}^{\beta}} g(u)\,du$, from Theorem 1.1.3.

(a) Let $Q_2 = \sum_{\ell=1}^{L} \overline{\int_{v_{\ell-1}}^{v_\ell}} g(u)\,du$. From Lemma 3.2.1, (A) $S(P) \ge S(\widehat{P}) = \sum_{\ell=1}^{L} S_\ell(P)$. From Theorem 1.1.8 and Definition 3.1.3, $S_\ell(P) \ge \overline{\int_{v_{\ell-1}}^{v_\ell}} g(u)\,du$. Therefore, (A) implies that $S(P) \ge Q_2$. Moreover, again from Definition 3.1.3 and Theorem 1.1.8, there is a a partition P_ℓ of $[v_{\ell-1}, v_\ell]$ such that $S(P_\ell) < \overline{\int_{v_{\ell-1}}^{v_\ell}} g(u)\,du + \frac{\epsilon}{L}$; hence, if P is the partition of $[\alpha, \beta]$ constructed from the points in $P_1, P_2, \dots, P_L$, then $S(P) < Q_2 + \epsilon$. Hence, $Q_2 = \overline{\int_{\alpha}^{\beta}} g(u)\,du$, from Theorem 1.1.8.

3.2.7. **(a)** Let V be the total variation of f on $[a, b]$. If $a < x < b$, then $f(x) = \frac{f(a) + f(b)}{2} + \frac{(f(x) - f(a)) + (f(x) - f(b))}{2}$; therefore,

$$|f(x)| \le \frac{|f(a) + f(b)|}{2} + \frac{|f(a) - f(x)| + |f(x) - f(b)|)}{2} \le \frac{|f(a) + f(b)| + V}{2}.$$

(b) Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ and $\epsilon > 0$. From Theorem 3.1.4, we can choose $c_1, \dots, c_n$ and $c'_1, \dots, c'_n$ so that $x_{j-1} \le c_j, c'_j \le x_j$, (A) $\left| S(P) - \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) \right| < \frac{\epsilon}{2}$ and (B) $\left| s(P) - \sum_{j=1}^{n} f(c'_j)(x_j - x_{j-1}) \right| < \frac{\epsilon}{2}$. Since $S(P) - s(P) = S(P) - \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) + \sum_{j=1}^{n} \left(f(c_j) - f(c'_j)\right)(x_j - x_{j-1}) s(P) + \sum_{j=1}^{n} f(c'_j)(x_j - x_{j-1})$, the triangle inequality, (A), and (B) imply that $S(P) - s(P) \le \epsilon + \sum_{j=1}^{n} |f(c_j) - f(c'_j)|(x_j - x_{j-1}) \le \epsilon + K\|P\| < 2\epsilon$ if $\|P\| < \frac{\epsilon}{K}$. Theorem 3.2.7 implies that f is integrable on $[a, b]$.

3.2.8.

$$\begin{aligned}\sum_{j=1}^{n} g(c_j)[f(x_j)-f(x_{j-1})] &= \sum_{j=1}^{n} g(c_j)f(x_j)-\sum_{j=1}^{n} g(c_j)f(x_{j-1})\\ &= \sum_{j=1}^{n} g(c_j)f(x_j)-\sum_{j=0}^{n-1} g(c_{j+1})f(x_j)\\ &= g(c_{n+1})f(x_n)-g(c_0)f(x_0)-\sum_{j=0}^{n} f(x_j)[g(c_{j+1})-g(c_j)]\\ &= g(b)f(b)-g(a)f(a)-\sum_{j=0}^{n} f(x_j)[g(c_{j+1})-g(c_j)].\end{aligned}$$

If $\sum_{j=1}^{n} g(c_j)[f(x_j)-f(x_{j-1})]$ is a Riemann-Stieltjes sum of g with respect to f over P, then $\sum_{j=0}^{n} f(x_j)[g(c_{j+1})-g(c_j)]$ is a Riemann-Stieltjes sum of f with respect to g over $P'=\{c_0,c_1,\dots,c_{n+1}\}$. (If $c_i=c_{i+1}$ for some i, then c_i is counted only once in P'.) Moreover, $\|P'\|\le 2\|P\|$. Now suppose $\epsilon>0$. Since $\int_a^b f(x)\,dg(x)$ exists, there is a $\delta>0$ such that $\left|\sum_{j=0}^{n} f(x_j)[g(c_{j+1})-g(c_j)]-\int_a^b f(x)\,dg(x)\right|<\epsilon$ if $\|P'\|<\delta$. Therefore, the identity derived above implies that

$$\left|\sum_{j=1}^{n} g(c_j)[f(x_j)-f(x_{j-1})]-f(b)g(b)+f(a)g(a)+\int_a^b f(x)\,dg(x)\right|<\epsilon$$

if $\|P\|<\delta/2$. This implies the conclusion.

3.2.9. **(a)** Let V be the total variation of g on $[a,b]$. Suppose that $\epsilon>0$. Choose $\delta>0$ so that (A) $|f(x)-f(x')|<\dfrac{\epsilon}{2V}$ if x and x' are in $[a,b]$ and $|x-x'|<\delta$ (Theorem 2.2.12). Let $P=\{x_0,x_1,\dots,x_n\}$ and $\sigma=\sum_{j=1}^{n} f(c_j)[g(x_j)-g(x_{j-1})]$. Let $P'=\{t_0,t_1,\dots,t_m\}$ be a refinement of P, and $\sigma'=\sum_{i=1}^{m} f(c_i')[g(t_i)-g(t_{i-1})]$. Suppose that $t_{k_j}=x_j$, $0\le j\le n$. Then (B) $|\sigma-\sigma'|\le\sum_{j=1}^{n}\sum_{i=1+k_{j-1}}^{k_j}|f(c_j)-f(c_i')|\,|g(t_i)-g(t_{i-1})|$. Since $x_{j-1}\le c_j\le x_j$ and $x_{j-1}\le c_i'\le x_j$ if $1+k_{j-i}\le i\le k_j$, it follows that $|c_j-c_i'|\le\|P\|$ if $1+k_{j-i}\le$

$i \le k_j$, $j = 1, \dots, n$. Therefore, (A) and (B) imply that $|\sigma - \sigma'| < \left(\frac{\epsilon}{2V}\right) V = \frac{\epsilon}{2}$ if $\|P\| < \delta$.

Let P_0 be a refinement of both P_1 and P_2, and let σ_0 be an Riemann-Stieltjes sum of f with respect to g over P_0. From **(a)**, $|\sigma_1 - \sigma_0| < \epsilon/2$ and $|\sigma_2 - \sigma_0| < \epsilon/2$. Therefore, $|\sigma_1 - \sigma_2| < \epsilon$, from the triangle inequality.

(c) Let $M = \sup_{a \le x \le b} f(x)$ (Theorem 2.2.8). If σ is any Riemann-Stieltjes sum of f with respect to g over $[a, b]$, then $|\sigma| \le MV$. Hence $|L(\rho)| \le MV$. Since $L(\delta)$ is nondecreasing and bounded below, $L = \inf_{\delta > 0} L(\delta)$ (Theorem 2.1.9).

(d) Suppose that$\epsilon > 0$. From **(a)** and **(c)** there is a $\delta > 0$ such that (C) $L \le L(\delta) < L + \epsilon$ and (D) $|\sigma - \sigma_0| < \epsilon$ if σ and σ_0 are Riemann-Stieltjes sums of f with respect to g over any partitions P and P_0 of $[a, b]$ with norm less than δ. From the definition of $L(\delta)$, we can choose P_0 and σ_0 so that $L(\delta) - \epsilon < \sigma_0 \le L(\delta)$. Then (C) implies that $|\sigma_0 - L| < \epsilon$. Now (D) and the triangle inequality imply that $|\sigma - L| \le |\sigma - \sigma_0| + |\sigma_0 - L| < 2\epsilon$ if σ is a Riemann-Stieltjes sum of f with respect to g over any partition P of $[a, b]$ with norm less than δ. Hence, $\int_a^b f(x)\, dg(x) = L$, by Definition 3.1.5.

3.2.10. $\int_a^b g(x)\, df(x)$ exists by Exercise 3.2.9 with f and g interchanged. Therefore, $\int_a^b f(x)\, dg(x)$ exists, by Exercise 3.2.8, again with f and g interchanged.

3.3 PROPERTIES OF THE INTEGRAL

3.3.1. Trivial if $c = 0$. Suppose $c \ne 0$ and $\epsilon > 0$. If $\widehat{\sigma}$ is a Riemann sum of cf, then $\widehat{\sigma} = \sum_{j=1}^{n} cf(c_j)(x_j - x_{j-1}) = c \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = c\sigma$, where σ is a Riemann sum for f. Since f is integrable on $[a, b]$, Definition 3.1.1 implies that there is a $\delta > 0$ such that $\left| \sigma - \int_a^b f(x)\, dx \right| < \frac{\epsilon}{|c|}$ if σ is a Riemann sum of f over any partition P of $[a, b]$ such that $\|P\| < \delta$. Therefore, $\left| \widehat{\sigma} - \int_a^b cf(x)\, dx \right| < \epsilon$ if $\widehat{\sigma}$ is a Riemann sum of cf over any partition P of $[a, b]$ such that $\|P\| < \delta$, so cf is integrable over $[a, b]$, again by Definition 3.1.1.

3.3.2. If f_1 and f_2 are integrable on $[a, b]$ and c_1 and c_2 are constants, then Theorem 3.3.2 implies that $c_1 f_1$ and $c_2 f_2$ are integrable on $[a, b]$ and $\int_a^b c_i f_i(x)\, dx = c_i \int_a^b f_i(x)\, dx$, $i = 1, 2$. Therefore, Theorem 3.3.1 implies P_2. Now suppose $n \ge 2$ and P_n is true. Let f_1, $f_2, \dots, f_{n+1}$ be integrable on $[a, b]$ and $c_1, c_2, \dots, c_{n+1}$ be constants. By Theorem 3.3.1, $c_1 f_1, c_2 f_2, \dots, c_{n+1} f_{n+1}$ are integrable on $[a, b]$, and $\int_a^b c_i f_i(x)\, dx = c_i \int_a^b f_i(x)\, dx$, $i = 1, 2, \dots, n+1$. Now $\int_a^b (c_1 f_1 + c_2 f_2 + \cdots + c_{n+1} f_{n+1})(x)\, dx = \int_a^b [(c_1 f_1 +$

$c_2 f_2 + \cdots + c_n f_n)(x) + c_{n+1} f_{n+1}(x)]\,dx = \int_a^b (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)(x)\,dx + \int_a^b c_{n+1} f_{n+1}(x)\,dx$ (by P_2)$= c_1 \int_a^b f_1(x)\,dx + c_2 \int_a^b f_2(x)\,dx + \cdots + c_n \int_a^b f_n(x)\,dx + c_{n+1} \int_a^b f_{n+1}(x)\,dx$ by P_n and Theorem 3.3.2. Therefore, P_n implies P_{n+1}.

3.3.3. Yes; let $f(x) = 1$ if x is rational, $f(x) = -1$ if x is irrational. Then $S(P) = b - a$ and $s(P) = a - b$ for every partition P of $[a, b]$, so f is not integrable on $[a, b]$. However, $|f(x)| = 1$ for all x in $[a, b]$, so $|f|$ is integrable on $[a, b]$.

3.3.4. If $f(x) \ge m_1$ and $g(x) \ge m_2$ $(a \le x \le b)$, write $fg = (f - m_1)(g - m_2) + m_2 f + m_1 g - m_1 m_2$. The first product on the right is integrable by the proof given in the text. To complete the proof, use Theorem 3.3.3.

3.3.5. Let "barred" quantities refer to $1/f$. First suppose $f(x) \ge \rho > 0$ $(a \le x \le b)$. If $P = \{x_0, \dots, x_n\}$ is a partition of $[a, b]$, then $\overline{M}_j = 1/m_j$ and $\overline{m}_j = 1/M_j$, so $\overline{M}_j - \overline{m}_j = (M_j - m_j)/m_j M_j \le (M_j - m_j)/\rho^2$; hence (A) $\overline{S}(P) - \overline{s}(P) < (S(P) - s(P))/\rho^2$. Now suppose $\epsilon > 0$. If f is integrable on $[a, b]$, then Theorem 3.2.7 implies that there is partition P of $[a, b]$ such that $S(P) - s(P) < \rho^2\epsilon$, so (A) implies that $\overline{S}(P) - \overline{s}(P) < \epsilon$. Therefore, $1/f$ is integrable on $[a, b]$, again by Theorem 3.2.7. Now suppose $|f(x)| \ge \rho > 0$ but $f(x) < 0$ for some x in $[a, b]$. Since f^2 is integrable on $[a, b]$ by Theorem 3.3.6, applying the result just proved to $g = f^2$ shows that $1/f^2$ is integrable on $[a, b]$. Therefore, $1/f = fg$ is integrable on $[a, b]$, again by Theorem 3.3.6.

3.3.6. Since $f^+ = (f + |f|)/2$, $f^- = (f - |f|)/2$ and $f = f^+ + f^-$, Theorems 3.3.3 and 3.3.5 imply the result.

3.3.7. **(a)** Since $\int_0^1 u(x)v(x)\,dx = \int_0^1 x^2\,dx = 1/3$ and $\int_0^1 v(x)\,dx = \int_0^1 x\,dx = 1/2$, $\overline{u} = c = 2/3$.

(b) Since $\int_{-1}^1 u(x)v(x)\,dx = \int_{-1}^1 x^2 \sin x\,dx = 0$, $\overline{u} = c = 0$.

(c) Since $\int_0^1 u(x)v(x)\,dx = \int_0^1 x^2 e^x\,dx = e - 2$ and $\int_0^1 v(x)\,dx = \int_0^1 e^x\,dx = e - 1$, $\overline{u} = (e - 2)/(e - 1)$ and $c = \sqrt{(e - 2)/(e - 1)}$.

3.3.8. Suppose $\epsilon > 0$. By Theorem 3.2.7 there are partitions P_1 of $[a, b]$ and P_2 of $[b, c]$ such that $S(P_1) - s(P_1) < \epsilon/2$ and $S(P_2) - s(P_2) < \epsilon/2$. Let P be the partition of $[a, c]$ with partition points from P_1 and P_2; then $S(P) - s(P) = [S(P_1) - s(P_1)] + [S(P_2) - s(P_2)] < \epsilon$. Hence f is integrable on $[a, c]$, by Theorem 3.2.7. To complete the proof, observe that Riemann sums over partitions of $[a, c]$ having b as a partition point can be broken up as $\sigma_{[a,c]} = \sigma_{[a,b]} + \sigma_{[b,c]}$.

3.3.9. If $a < b < c$, Theorem 3.3.9 implies the conclusion. There are eight other possible orderings of a, b, and c. Suppose, for example, that $c < a < b$. Then Theorem 3.3.9 implies that (A) $\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx$. Since $\int_c^b f(x)\,dx = -\int_b^c f(x)\,dx$ and $\int_c^a f(x)\,dx = -\int_a^c f(x)\,dx$ by definition, (A) is equivalent to $-\int_b^c f(x)\,dx = -\int_a^c f(x)\,dx + \int_a^b f(x)\,dx$, which is equivalent to $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$. The other possibilities can be handled similarly.

3.3.10. Proof by induction. Theorems 3.3.8 and 3.3.9 imply P_2. Now suppose $n \ge 2$ and P_n is true. Let $a = a_0 < a_1 < \cdots < a_{n+1} = b$. Theorem 3.3.8 and P_2 imply that (A) $\int_a^b f(x)\,dx = \int_{a_0}^{a_n} f(x)\,dx + \int_{a_n}^{a_{n+1}} f(x)\,dx$. By P_n, $\int_{a_0}^{a_n} f(x)\,dx = \int_{a_0}^{a_1} f(x)\,dx +$

$\cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx$. This and (A) imply P_{n+1}

3.3.11. By Exercise 3.3.10, (A) $\int_a^b f(x)\,dx = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)\,dx$. Applying Theorem 3.3.7 with $u = f$, $v = 1$, $a = x_{j-1}$, and $b = x_j$ shows that $\int_{x_{j-1}}^{x_j} f(x)\,dx = f(\overline{c}_j)(x_j - x_{j-1})$ for some $\overline{c}_j \in [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$. Therefore, (A) implies that $\int_a^b f(x)\,dx = \sum_{j=1}^n f(\overline{c}_j)(x_j - x_{j-1})$.

3.3.12. Let $P = \{x_0, x_1, \dots, x_n\}$ and let $\sigma_0 = \sum_{j=1}^n f(\overline{c}_j)(x_j - x_{j-1})$ be the Riemann sum chosen in the solution of Exercise 3.3.11. If $\sigma = \sum_{j=1}^n f(c_j)(x_j - x_{j-1})$ is any Riemann sum over P, then (A) $\left|\sigma - \int_a^b f(x)\,dx\right| = |\sigma - \sigma_0| \le \sum_{j=1}^n |f(c_j) - f(\overline{c}_j)|(x_j - x_{j-1})$. From Theorem 2.3.14, $|f(c_j) - f(\overline{c}_j)| \le M(x_j - x_{j-1}) \le M\|P\|$. Therefore, (A) implies that $\left|\sigma - \int_a^b f(x)\,dx\right| \le M\|P\| \sum_{j=1}^n (x_j - x_{j-1}) = M\|P\|(b-a)$.

3.3.13. **(a)** $\int_a^b f(x)\,dx \ge 0$ because every Riemann sum for f is ≥ 0. If f is continuous at c in $[a, b]$ and $f(c) > 0$, there is an interval $[\alpha, \beta] \subset [a, b]$ such that $f(x) > f(c)/2$ if $x \in [\alpha, \beta]$. From Exercise 3.3.9, (A) $\int_a^b f(x)\,dx = \int_a^\alpha f(x)\,dx + \int_\alpha^\beta f(x)\,dx + \int_\beta^b f(x)\,dx$. Since $\int_a^\alpha f(x)\,dx \ge 0$ and $\int_\beta^b f(x)\,dx \ge 0$, (A) implies that $\int_a^b f(x)\,dx \ge \int_\alpha^\beta f(x)\,dx \ge f(c)(\beta - \alpha)/2$, where the last inequality follows from Theorem 3.3.2.

3.3.14. Since $\dfrac{1}{x-a}\displaystyle\int_a^x f(a)\,dt = f(a)$, we can write

$$\frac{F(x) - F(a)}{x-a} - f(a) = \frac{1}{x-a}\int_a^x [f(t) - f(a)]\,dt.$$

From this and Theorem 3.3.5, (A) $\left|\dfrac{F(x) - F(a)}{x-a} - f(a)\right| \le \dfrac{1}{|x-a|}\left|\displaystyle\int_a^x |f(t) - f(a)|\,dt\right|$. Since f is continuous from the right at a, there is for each $\epsilon > 0$ a $\delta > 0$ such that $|f(t) - f(a)| < \epsilon$ if $a \le x < a + \delta$ and t is between x and a. Therefore, from (A), $\left|\dfrac{F(x) - F(a)}{x-a} - f(a)\right| < \epsilon\dfrac{|x-a|}{|x-a|} = \epsilon$ if $a < x < a + \delta$. This proves that $F'_+(a) = f(a)$.

Since $\dfrac{1}{b-x}\displaystyle\int_x^b f(b)\,dt = f(b)$, we can write

$$\frac{F(x) - F(b)}{b-x} - f(b) = \frac{1}{b-x}\int_x^b [f(b) - f(t)]\,dt.$$

From this and Theorem 3.3.5, (B) $\left|\dfrac{F(x) - F(b)}{b-x} - f(b)\right| \le \dfrac{1}{|b-x|}\left|\displaystyle\int_x^b |f(b) - f(t)|\,dt\right|$. Since f is continuous from the left at b, there is for each $\epsilon > 0$ a $\delta > 0$ such that $|f(b) - f(t)| < \epsilon$ if $b - \delta \le x < b$ and t is between x and b. Therefore, from (B),

$\left|\dfrac{F(x)-F(b)}{b-x}-f(b)\right| < \epsilon\dfrac{|b-x|}{|b-x|} = \epsilon$ if $b-\delta < x < b$. This proves that $F'_{-}(b) = f(b)$.

3.3.15. THEOREM. *If f is integrable on $[a,b]$ and $a \le c \le b$, then the function g defined by $G(x) = \int_x^c f(t)\,dt$ satisfies a Lipschitz condition on $[a,b]$, and is therefore continuous on $[a,b]$.*

THEOREM. *If f is integrable on $[a,b]$ and $a \le c \le b$, then $G(x) = \int_x^c f(t)\,dt$ is differentiable at any point in (a,b) where f is continuous, with $G'(x_0) = -f(x_0)$. If f is continuous from the right at a, then $G'_{+}(a) = -f(a)$. If f is continuous from the left at b, then $G'_{-}(b) = -f(b)$.*

Rewrite $G(x) = \int_c^x(-f(t))\,dt$ and apply Theorems 3.3.10 and 3.3.11 to $g = -f$.

3.3.16. **(a)** Since arbitrary constants can be added to antiderivatives, (A) need not hold for specific antiderivatives of f, g, and $f+g$.

(b) Every antiderivative of $f+g$ can be obtained by adding antiderivatives of f and g, and every such sum is an antiderivative of $f+g$.

3.3.17. If $c \ne 0$, every antiderivative of cf can be written as cF, where F is an antiderivative of f; conversely, if F is an antiderivative of f, then cF is an antiderivative of cf. This statement is false if $c = 0$, since then any constant is an antiderivative of cf, while $cF = 0$ for any F.

3.3.18. **(a)** P_0 is true, by Corollary 3.3.13. Now suppose $n \ge 0$ and P_n is true. If $f^{(n+2)}$ is integrable on $[a,b]$, integration by parts yields

$$\int_a^b f^{(n+1)}(t)(b-t)^n\,dt = \frac{f^{(n+1)}(a)}{n+1}(b-a)^{n+1} + \frac{1}{n+1}\int_a^b f^{(n+2)}(t)(b-t)^{n+1}\,dt.$$

Now P_n implies P_{n+1}.

(b) If $f^{(n+1)}$ is continuous on $[a,b]$,**(a)** and Theorem 3.3.7 imply the conclusion of Theorem 2.5.5.

3.3.19. If c and c_1 have the property, then $f(b)\int_c^{c_1} g(x)\,dx = 0$, which implies that $\int_c^{c_1} g(x)\,dx = 0$, since $f(b) > 0$. Exercise 3.3.13 implies that $c = c_1$.

3.3.20. Let $m = \inf\{f(x) \mid a \le x \le b\}$ and define $F(x) = f(x) - m$. Then F is nonnegative on $[a,b]$. Moreover, F is integrable on $[a,b]$ if and only if f is, while $F(\phi(t))\phi'(t)$ is integrable on $[c,d]$ if and only if $f(\phi(t))\phi'(t)$ is. The assumed version of Theorem 3.3.18 implies that if either F is integrable on $[a,b]$ or $F(\phi(t))\phi'(t)$ is integrable on $[c,d]$ then so is the other, and (A) $\int_a^b F(x)\,dx = \int_c^d F(\phi(t))|\phi'(t)|\,dt$. Now note that (B) $\int_a^b m\,dx = (b-a)$. If ϕ is nondecreasing, then $\phi(c) = a$, $\phi(d) = b$, and $|\phi'(t)| = \phi'(t)$, so (C) $\int_c^d m|\phi'(t)|\,dt = \int_c^d m\phi'(t)\,dt = m(\phi(d)-\phi(c)) = m(b-a)$. If ϕ is nonincreasing, then $\phi(c) = b$, $\phi(d) = a$, and $|\phi'(t)| = -\phi'(t)$, so (D) $\int_c^d m|\phi'(t)|\,dt = -\int_c^d m\phi'(t)\,dt = -m(\phi(d)-\phi(c)) = m(b-a)$. Now (A), (B), (C), and (D) imply that if either f is integrable on $[a,b]$ or $f(\phi(t))\phi'(t)$ is integrable on $[c,d]$ then so is the other, and $\int_a^b f(x)\,dx = \int_c^d f(\phi(t))|\phi'(t)|\,dt$.

3.3.21. If ϕ is nonincreasing on $[c,d]$, then $\phi(-t)$ is nondecreasing on $[-d,-c]$, and (A) $\frac{d}{dt}\phi(-t) = -\phi'(-t)$. Now suppose f is integrable on $[a,b]$. Then Exercise 3.1.6 implies that $f(-x)$ is integrable on $[-b,-a]$, and (B) $\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx$. Applying the assumed restricted version of Theorem 3.3.18 to $\phi(-t)$ and $f(-x)$ and recalling (A), we infer that (C) $\int_{-b}^{-a} f(-x)\,dx = -\int_{-d}^{-c} f(\phi(-t))\phi'(-t)\,dt$. Applying Exercise 3.1.6 to $f(\phi(t))\phi'(t)$, we conclude that (D) $\int_{-d}^{-c} f(\phi(-t))\phi'(-t)\,dt = \int_c^d f(\phi(t))\phi'(t)\,dt$. Now (B), (C), and (D) imply that (E) $\int_a^b f(x)\,dx = -\int_c^d f(\phi(t))\phi'(t)\,dt$. Since $|\phi'(t)| = -\phi'(t)$, (E) is equivalent to $\int_a^b f(x)\,dx = \int_c^d f(\phi(t))|\phi'(t)|\,dt$. A similar argument applies if we start with the assumption that $f(\phi(t))\phi'(t)$ is integrable on $[c,d]$.

3.3.22. A typical RS sum for $\int_a^b f(x)\,dg(x)$ is

$$\sigma = \sum_{j=1}^n f(c_j)[g(x_j) - g(x_{j-1})] = \sum_{j=1}^n f(c_j)g'(\overline{c}_j)(x_j - x_{j-1})$$

where $x_{j-1} < c_j < x_j$ (mean value theorem). Let $\sigma' = \sum_{j=1}^n f(\overline{c}_j)g'(\overline{c}_j)(x_j - x_{j-1})$, which is a Riemann sum for $\int_a^b f(x)g'(x)\,dx$. Then

$$|\sigma - \sigma'| \le \left(\max\left\{|f(c_j) - f(\overline{c}_j)| \,\middle|\, 1 \le j \le n\right\}\right) M(b-a),$$

where $|g'(x)| \le M$ on $[a,b]$. By Theorem 2.2.12, f is uniformly continuous on $[a,b]$; therefore, $|\sigma - \sigma'|$ can be made arbitrarily small by choosing $\|P\|$ sufficiently small. Since

$$\left|\sigma - \int_a^b f(x)g'(x)\,dx\right| \le |\sigma - \sigma'| + \left|\sigma' - \int_a^b f(x)g'(x)\,dx\right|,$$

the existence of $\int_a^b f(x)g'(x)\,dx$ implies the conclusion.

3.3.23. A typical RS sum for $\int_a^b f(x)\,dg(x)$ is (A) $\sigma = \sum_{j=1}^n f(c_j)[g(x_j) - g(x_{j-1})]$. From Theorem 2.5.4, (B) $g(x_j) = g(c_j) + g'(c_j)(x_j - c_j) + \frac{g''(\alpha_j)}{2}(x_j - c_j)^2$ for some $\alpha_j \in (c_j, x_j)$ and (C) $g(x_{j-1}) = g(c_j) + g'(c_j)(x_{j-1} - c_j) + \frac{g''(\alpha_{j-1})}{2}(x_{j-1} - c_j)^2$ for some $\alpha_{j-1} \in (x_{j-1}, c_j)$. Subtracting (C) from (B) and substituting the result in (A) yields

$$\sigma = \sigma' + \frac{1}{2}\sum_{j=1}^n f(c_j)[g''(\alpha_j)(x_j - c_j)^2 - g''(\alpha_{j-1})(x_{j-1} - c_j)^2], \qquad \text{(D)}$$

where $\sigma' = \sum_{j=1}^{n} f(c_j)g'(c_j)(x_j - x_{j-1})$ is a Riemann sum for $\int_a^b f(x)g'(x)\,dx$. Now suppose $|f(x)| \le M$ and $|g''(x)| \le K$, $a \le x \le b$. From (D),

$$|\sigma - \sigma'| \le \frac{MK}{2} \sum_{j=1}^{n} \left[(x_j - c_j)^2 + (x_{j-1} - c_j)^2\right] \le MK(b-a)\|P\|.$$

Therefore,

$$\begin{aligned}\left|\sigma - \int_a^b f(x)g'(x)\,dx\right| &\le |\sigma - \sigma'| + \left|\sigma' - \int_a^b f(x)g'(x)\,dx\right| \\ &\le MK(b-a)\|P\| + \left|\sigma' - \int_a^b f(x)g'(x)\,dx\right|.\end{aligned} \tag{E}$$

If $\epsilon > 0$, there is a $\delta > 0$ such that $MK(b-a)\delta < \frac{\epsilon}{2}$ and $\left|\sigma' - \int_a^b f(x)g'(x)\,dx\right| < \frac{\epsilon}{2}$ if $\|P\| < \delta$. Therefore, (E) implies that $\left|\sigma - \int_a^b f(x)g'(x)\,dx\right| < \epsilon$ if $\|P\| < \delta$. This implies that $\int_a^b f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx$.

3.4 IMPROPER INTEGRALS

3.4.1. **(a)** Suppose that $|f(x)| \le M$, $a \le x \le b$ after $f(b)$ is defined. If $\epsilon > 0$, choose b_1 so that $a < b_1 < b$ and $M(b - b_1) < \epsilon$. Since f is integrable on $[a, b_1]$, there is a partition $P_1 = \{a, x_1, \dots, b_1\}$ of $[a, b_1]$ such that $S(P_1) - s(P_2) < \epsilon$ (Theorem 3.2.7). Then $P = \{a, x_1, \dots, b_1, b\}$ is a partition of $[a, b]$ for which $S(P) - s(P) < 3\epsilon$. Hence, $\int_a^b f(x)\,dx$ exists (Theorem 3.2.7). Now define $F(c) = \int_a^c f(x)\,dx$ $(a \le c \le b)$. Since F is continuous on $[a, b]$ (Theorem 3.3.10), $F(b-) = F(b)$, which implies that $\int_a^b f(x)\,dx = \lim_{c\to b-} \int_a^c f(x)\,dx$. Since $\lim_{c\to b-} \int_a^c f(x)\,dx$ is independent of $f(b)$, $\int_a^b f(x)\,dx$ is independent of $f(b)$.

(b) Let f be locally integrable and bounded on $(a, b]$, and let $f(a)$ be defined arbitrarily. Then f is properly integrable on $[a, b]$, $\int_a^b f(x)\,dx$ does not depend on $f(a)$, and $\int_a^b f(x)\,dx = \lim_{c\to a+} \int_c^b f(x)\,dx$.

3.4.2. Let α and α_1 be in (a, b). Suppose that we know that the improper integrals $\int_a^\alpha f(x)\,dx$ and $\int_\alpha^b f(x)\,dx$ both exist. Let $a < c < b$. Then $\int_c^{\alpha_1} f(x)\,dx = \int_c^{\alpha} f(x)\,dx +$

$\int_{\alpha}^{\alpha_1} f(x)\,dx$. Letting $c \to a+$ shows that (A) $\int_{a}^{\alpha_1} f(x)\,dx = \int_{a}^{\alpha} f(x)\,dx + \int_{\alpha}^{\alpha_1} f(x)\,dx$. Also, $\int_{\alpha_1}^{c} f(x)\,dx = \int_{\alpha_1}^{\alpha} f(x)\,dx + \int_{\alpha}^{c} f(x)\,dx$. Letting $c \to b-$ shows that (B) $\int_{\alpha_1}^{b} f(x)\,dx = \int_{\alpha_1}^{\alpha} f(x)\,dx + \int_{\alpha}^{b} f(x)\,dx$. Adding (A) and (B) and noting that $\int_{\alpha}^{\alpha_1} f(x)\,dx = -\int_{\alpha_1}^{\alpha} f(x)\,dx$ shows that $\int_{a}^{\alpha} f(x)\,dx + \int_{\alpha}^{b} f(x)\,dx = \int_{a}^{\alpha_1} f(x)\,dx + \int_{\alpha_1}^{b} f(x)\,dx$.

3.4.3. Suppose that f is locally integrable on $[a,b)$ and $\int_{a}^{b} f(x)\,dx$ exists according to Definition 3.4.1. Let $a < \alpha < c$. Then $\int_{\alpha}^{c} f(x)\,dx = \int_{\alpha}^{a} f(x)\,dx + \int_{a}^{c} f(x)\,dx$, so $\lim_{c\to b-} \int_{\alpha}^{c} f(x)\,dx$ exists. Theorem 3.3.8 implies that $\int_{a}^{\alpha} f(x)\,dx$ exists (as a proper integral), and Theorem 3.3.10 implies that $\lim_{c\to a+} \int_{c}^{\alpha} f(x)\,dx = \int_{a}^{\alpha} f(x)\,dx$. Hence, $\int_{a}^{b} f(x)\,dx$ exists according to Definition 3.4.3. Moreover, $\int_{a}^{b} f(x)\,dx = \int_{a}^{\alpha} f(x)\,dx + \lim_{c\to b-} \int_{\alpha}^{c} f(x)\,dx = \lim_{c\to b-} \left(\int_{a}^{\alpha} f(x)\,dx + \int_{\alpha}^{c} f(x)\,dx \right) = \lim_{c\to b-} \int_{a}^{c} f(x)\,dx$, so Definitions 3.4.1 and 3.4.3 yield the same value for $\int_{a}^{b} f(x)\,dx$. A similar argument applies if f is locally integrable on $(a,b]$ and $\int_{a}^{b} f(x)\,dx$ exists according to Definition 3.4.2.

3.4.4. **(a)** Proper if $p \ge 2$, improper if $p < 2$. In either case

$$I = \lim_{c\to 0+} \int_{c}^{1/\pi} \frac{d}{dx}\left(x^p \sin\frac{1}{x} \right) dx = -c^p \sin\frac{1}{c} = 0$$

if $p > 0$; divergent if $p < 0$.

(b) Proper if $p \ge 2$, improper if $p < 2$. In either case

$$I = \lim_{c\to 0+} \int_{c}^{2/\pi} \frac{d}{dx}\left(x^p \cos\frac{1}{x} \right) dx = -c^p \cos\frac{1}{c} = 0$$

if $p > 0$; divergent if $p < 0$.

(c) Improper for all p. $\int_{0}^{c} e^{-px}\,dx = \begin{cases} (1-e^{-pc})/p, & p \ne 0, \\ c, & p = 0. \end{cases}$ Therefore,

$$\int_{0}^{\infty} e^{-px}\,dx = \begin{cases} 1/p, & p > 0, \\ \infty, & p \le 0. \end{cases}$$

(d) Proper for $p \le 0$, improper for $p > 0$. $\int_{c}^{1} x^{-p}\,dx = \begin{cases} (1-c^{-p+1})/(1-p), & p \ne 0, \\ -\log c, & p = 1. \end{cases}$

Therefore, $\displaystyle\int_0^1 x^{-p}\,dx = \begin{cases} 1/(1-p), & p<1, \\ \infty, & p \ge 1 \end{cases}$

(e) Improper for all p. Write the integral as $I = I_1 + I_2$, with $I_1 = \displaystyle\int_0^1 x^{-p}\,dx$ and $I_2 = \displaystyle\int_1^\infty x^{-p}\,dx$. From **(d)**, I_1 converges if and only if $p<1$. Now consider I_2. $\displaystyle\int_1^c x^{-p}\,dx = \begin{cases} (c^{-p+1}-1)/(1-p), & p \ne 1, \\ \log c, & p = 1. \end{cases}$ Therefore, $I_2 = \begin{cases} 1/(p-1), & p>1, \\ \infty, & p \le 1. \end{cases}$ Since I_1 converges only if $p<1$ and I_2 converges only if $p>1$, I diverges for all p.

3.4.5. **(a)** We show by induction that $\displaystyle\int_0^\infty x^n e^{-x}\,dx = n!$. $\displaystyle\int_0^c e^{-x}\,dx = 1-e^{-c}$. Letting $c\to 0$ verifies P_0. Now let $n \ge 0$ and suppose that P_n is true. Integrating by parts yields $\displaystyle\int_0^c e^{-x}x^{n+1}\,dx = -e^{-c}c^{n+1} + (n+1)\int_0^c e^{-x}x^n\,dx$. Letting $c\to\infty$ and invoking P_n yields $\displaystyle\int_0^\infty e^{-x}\,dx = (n+1)n! = (n+1)!$.

(b) $\displaystyle\int_0^c e^{-x}\sin x\,dx = -\frac{e^{-c}}{2}(\cos c + \sin c) + \frac{1}{2}$, so $\displaystyle\int_0^\infty e^{-x}\sin x\,dx = \frac{1}{2}$.

(c) Write $I = I_1 + I_2$ with $I_1 = \displaystyle\int_0^c \frac{x\,dx}{x^2+1}$ and $I_2 = \displaystyle\int_{-c}^0 \frac{x\,dx}{x^2+1}$. Since $I_1 = \dfrac{1}{2}\log(c^2+1) \to \infty$ as $c\to\infty$, I_1 diverges. Therefore, I diverges.

(d) $\displaystyle\int_0^c \frac{x\,dx}{\sqrt{1-x^2}} = 1-\sqrt{1-c^2} \to 1$ as $c \to 1-$, so $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = 1$.

(e) $\displaystyle\int_c^\pi \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)dx = -\frac{\sin c}{c} \to -1$ as $c \to 0+$. Therefore, $\displaystyle\int_0^\pi \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)dx = -1$.

(f) $\displaystyle\int_{\pi/2}^c \left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)dx = -\frac{\cos c}{c} \to 0$ as $c\to\infty$. Therefore, $\displaystyle\int_{\pi/2}^\infty \left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)dx = 0$.

3.4.6. If $\epsilon > 0$, there is an a_1 such that $a < a_1 < b$ and $|\int_a^c f(t)\,dt - \int_a^b f(t)\,dt| < \epsilon/2$ if $a_1 < c < b$ (from Definition 3.4.1 if $\displaystyle\int_a^b f(t)\,dt$ is improper; from Exercise 3.4.1 if it is proper). Therefore, if $a_1 < x < c < b$,

$$\begin{aligned}\left|\int_x^c f(t)\,dt\right| &= \left|\int_a^c f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &\le \left|\int_a^c f(t)\,dt - \int_a^b f(t)\,dt\right| + \left|\int_a^b f(t)\,dt - \int_a^x f(t)\,dt\right| < \frac{\epsilon}{2}.\end{aligned}$$

Letting $c \to b-$ here shows that $\left|\int_x^b f(x)\,dx\right| \le \epsilon$ if $a_1 < x < b$. This implies the stated conclusion.

3.4.7. Apply Exercise 2.1.38 to $F(x) = \int_a^x f(t)\,dt$.

3.4.8. **(a)** $\dfrac{\log x + \sin x}{\sqrt{x}} > \dfrac{1}{\sqrt{x}}$ if $x > e^2$. Since $\int_{e^2}^\infty \dfrac{dx}{\sqrt{x}} = \infty$, $\int_{e^2}^\infty \dfrac{\log x + \sin x}{\sqrt{x}}\,dx = \infty$. Therefore, $\int_1^\infty \dfrac{\log x + \sin x}{\sqrt{x}}\,dx = \infty$ (Theorem 3.4.6(**b**)).

(b) $I = I_1 + I_2$ where $I_1 = \int_0^\infty \dfrac{(x^2+3)^{3/2}}{(x^4+1)^{3/2}} \sin^2 x\,dx$ and $I_2 = \int_{-\infty}^0 \dfrac{(x^2+3)^{3/2}}{(x^4+1)^{3/2}} \sin^2 x\,dx$. $\dfrac{(x^2+3)^{3/2}}{(x^4+1)^{3/2}} \sin^2 x < \dfrac{(x^2+3)^{3/2}}{(x^4+1)^{3/2}} < \dfrac{(x^2+3)^{3/2}}{x^6} < \dfrac{1}{|x|^3}\left(1+\dfrac{3}{x^2}\right)^{3/2} < \dfrac{2^{3/2}}{|x|^3}$ if $|x| > \sqrt{3}$. Since $\int_{\sqrt{3}}^\infty \dfrac{1}{x^3}\,dx < \infty$ and $\int_{-\infty}^{-\sqrt{3}} \dfrac{1}{|x|^3}\,dx < \infty$, $\int_{\sqrt{3}}^\infty \dfrac{(x^2+3)^{3/2}}{(x^4+1)^{3/2}} \sin^2 x\,dx < \infty$ and $\int_{-\infty}^{-\sqrt{3}} \dfrac{(x^2+3)^{3/2}}{(x^4+1)^{3/2}} \sin^2 x\,dx < \infty$. Therfore I_1 and I_2 converge (Theorem 3.4.6(**a**)), and so does I.

(c) $\dfrac{1+\cos^2 x}{\sqrt{1+x^2}} > \dfrac{1}{\sqrt{1+x^2}} = \dfrac{1}{x\sqrt{1+1/x^2}} > \dfrac{1}{x\sqrt{2}}$ if $x > 1$. Since $\int_1^\infty \dfrac{dx}{x} = \infty$, $\int_0^\infty \dfrac{1+\cos^2 x}{\sqrt{1+x^2}}\,dx = \infty$ (Theorem 3.4.6(**b**)).

(d) $I = I_1 + I_2$ where $I_1 = \int_0^1 \dfrac{4+\cos x}{(1+x)\sqrt{x}}\,dx$ and $I_2 = \int_1^\infty \dfrac{4+\cos x}{(1+x)\sqrt{x}}\,dx$. $0 < \dfrac{4+\cos x}{(1+x)\sqrt{x}} < \dfrac{5}{\sqrt{x}}$ if $x > 0$. Since $\int_0^1 \dfrac{dx}{\sqrt{x}}$, $I_1 < \infty$ (Theorem 3.4.6(**a**)). $0 < \dfrac{4+\cos x}{(1+x)\sqrt{x}} < \dfrac{5}{x^{3/2}}$ if $x > 1$. Since $\int_0^1 \dfrac{dx}{x^{3/2}}$, $I_2 < \infty$ (Theorem 3.4.6(**a**)). Therefore, I converges.

(e) Convergent, by Exercise 3.4.5(**a**),(**b**), and Theorem 3.4.4.

(f) Since $2+\sin x > 1$ and $\int_0^\infty x^{-p}\,dx = \infty$ (Exercise 3.4.4(**a**)), $\int_0^\infty x^{-p}(2+\sin x)\,dx = \infty$

3.4.9. **(a)** $\lim_{x\to 0+} \left(\dfrac{\sin x}{x^p}\right) \Big/ \left(\dfrac{1}{x^{p-1}}\right) = \lim_{x\to 0+} \dfrac{\sin x}{x} = 1$, so the analog of Theorem 3.4.7(**a**) implies that $I = \int_0^{\pi/2} \dfrac{\sin x}{x^p}\,dx$ and $I_1 = \int_0^{\pi/2} \dfrac{dx}{x^{p-1}}$ converge or diverge together. Since $\int_c^{\pi/2} \dfrac{dx}{x^{p-1}} = \dfrac{1}{2-p}\left[\left(\dfrac{\pi}{2}\right)^{-p+2} - c^{-p+2}\right]$, $I_1 < \infty$–and therefore $I < \infty$–if and only if $p < 2$.

(b) $\lim_{x\to 0+}\left(\frac{\cos x}{x^p}\right)\Big/\left(\frac{1}{x^p}\right) = \lim_{x\to 0+}\cos x = 1$, so the analog of Theorem 3.4.7**(a)** implies that $I = \int_0^{\pi/2}\frac{\cos x}{x^p}\,dx$ and $I_1 = \int_0^{\pi/2}\frac{dx}{x^p}$ converge or diverge together. Since $\int_c^{\pi/2}\frac{dx}{x^p} = \frac{1}{1-p}\left[\left(\frac{\pi}{2}\right)^{-p+1} - c^{-p+1}\right]$, $I_1 < \infty$–and therefore $I < \infty$–if and only if $p < 1$.

(c) Write $I = I_1 + I_2$ where $I_1 = \int_0^1 x^p e^{-x}\,dx$ and $I_2 = \int_1^\infty x^p e^{-x}\,dx$. Since $\lim_{x\to 0+}(x^p e^{-x})/x^p = \lim_{x\to 0+} e^{-x} = 1$, $I_1 < \infty$ if and only if (A) $\int_0^1 x^p\,dx < \infty$, by the analog of Theorem 3.4.7**(a)**. Since (A) holds if and only if $p > -1$, $I_1 < \infty$ if and only if $p > -1$. Since $\lim_{x\to\infty}(x^{-p}e^{-x})/e^{-x/2} = \lim_{x\to\infty} x^p e^{-x/2} = 0$ for all p and $\int_1^\infty e^{-x/2}\,dx < \infty$, the analog of Theorem 3.4.7**(a)** implies that $I_2 < \infty$ for all p. Therefore, $I < \infty$ if and only if $p > -1$

(d) Let $f(x) = \sin x(\tan x)^{-p} = (\cos x)^p(\sin x)^{-p+1}$. Then $I = \int_0^{\pi/2} f(x)\,dx$ is a proper integral if $0 \le p \le 1$.

If $p < 0$, then f is locally integrable on $[0, \pi/2)$. The mean value theorem implies that $\cos x = -(\sin c)(x - \pi/2)$ for some c between x and $\pi/2$. Therefore, $\lim_{x\to\pi/2}\frac{\cos x}{x-\pi/2} = 1$, so $\lim_{x\to\pi/2-}\frac{f(x)}{(x-\pi/2)^p} = 1$. Since $\int_0^{\pi/2}(x-\pi/2)^p\,dx < \infty$ if and only if $p > -1$, Theorem 3.4.7**(a)** implies that $I < \infty$ if $-1 < p \le 0$ and $I = \infty$ if $p \le -1$.

If $p > 1$, then f is locally integrable on $(0, \pi/2]$. The mean value theorem implies that $\cos x = -(\sin c)(x - \pi/2)$ for some c between x and $\pi/2$. Therefore, $\lim_{x\to\pi/2}\frac{\cos x}{x-\pi/2} = 1$, so $\lim_{x\to\pi/2-}\frac{f(x)}{(x-\pi/2)^p} = 1$. Since $\int_0^{\pi/2}(x-\pi/2)^p\,dx < \infty$ if and only if $p > -1$, Theorem 3.4.7**(a)** implies that $I < \infty$ if $-1 < p \le 0$ and $I = \infty$ if $p \le -1$.

3.4.10. The substitution $u = L_{k+1}(x)$ transforms $\int_a^T f(x)\,dx$ into $\int_{L_{k+1}(a)}^{L_{k+1}(T)} u^{-p}\,du$. Since $\lim_{T\to\infty} L_{k+1}(T) = \infty$, Example 3.4.3 yields the conclusion.

3.4.12. If $f(x) = a_0 + \cdots + a_n x^n$ and $g(x) = b_0 + \cdots + b_m x^m$ ($a_n, b_m \ne 0$), then

$$\lim_{x\to\pm\infty}\frac{\left|\frac{f(x)}{g(x)}\right|}{|x|^{n-m}} = |a_n/b_m|.$$

Therefore, $\int_1^\infty |\frac{f(x)}{g(x)}|dx < \infty$ and $\int_{-\infty}^1 |\frac{f(x)}{g(x)}|dx < \infty$ if and only if $\int_1^\infty x^{n-m}dx < \infty$; that is, if and only if $m - n \ge 2$.

3.4.13. $(f+g)^2 = f^2 + 2fg + g^2 \ge 0$, so $fg \ge -(f^2+g^2)/2$; $(f-g)^2 = f^2 - 2fg + g^2 \ge 0$, so $fg \le (f^2+g^2)/2$. Therefore, $|fg| \le (f^2+g^2)/2$. Now use

Theorem 3.4.6.

3.4.14. Since f is nonoscillatory at $b-$, there is an interval $[a_1, b)$ on which f does not change sign. Assume that $f(x) \ge 0$ and $|g(x)| \le M$ for $a_1 \le x < b$; then

$$\int_{a_1}^{c} |f(x)g(x)|dx < M \int_{a_1}^{c} f(x)dx = M\left[F(c) - F(a_1)\right].$$

Since F is bounded on $[a, b)$, this implies that $\sup_{a_1 \le c < b} \int_{a_1}^{c} |f(x)g(x)|dx < \infty$. Use Theorem 3.4.5. A similar argument applies if $f(x) \le 0$ on $[a_1, b)$.

3.4.15. Since g is nonincreasing, it is locally integrable on $[a, b)$ (Theorem 3.2.9), and therefore so is fg (Theorem 3.3.6). Since $|f(x)g(x)| \le g(a)|f(x)|$, $\int_a^b |f(x)g(x)|dx < \infty$ if $\int_a^b |f(x)|dx < \infty$ (Theorem 3.4.6); hence, $\int_a^b f(x)g(x)dx$ converges (Theorem 3.4.9). If $a < x < c < b$, then $\left|\int_x^c f(t)g(t)dt\right| \le \int_x^c |f(t)|g(t)dt \le g(x)\int_x^c |f(t)|dt$. Letting $c \to b-$ yields $\left|\int_x^b f(t)g(t)dt\right| \le g(x)\int_x^b |f(t)|dt$. Now divide by $g(x)$ and apply Exercise 3.4.6 to $|f|$.

3.4.16. If f does not change sign on $[a_1, b)$, where $a < a_1 < b$, then obviously $\int_{a_1}^b f(x)dx$ and $\int_{a_1}^b |f(x)|dx$ converge or diverge together. Since $\int_a^b f(x)dx$ and $\int_{a_1}^b f(x)dx$ converge or diverge together, as do $\int_a^b |f(x)|dx$ and $\int_{a_1}^b |f(x)|dx$, the conclusion follows.

3.4.17. **(a)** If g is nonincreasing, then $g(x) \ge 0$ on $[a, b)$ (since $\int_a^b g(x)\,dx = \infty$) and

$$\begin{aligned}
\int_a^{x_{r+1}} |f(x)g(x)|\,dx &\ge \sum_{j=0}^{r} \int_{x_j}^{x_{j+1}} |f(x)|\,g(x)\,dx \ge \rho \sum_{j=0}^{r} g(x_{j+1}) \\
&\ge \rho \sum_{j=0}^{r} \frac{\int_{x_{j+1}}^{x_{j+2}} g(x)\,dx}{x_{j+2} - x_{j+1}} \\
&\ge \frac{\rho}{M} \int_{x_1}^{x_{r+2}} g(x)\,dx \to \infty \qquad \text{as} \qquad r \to \infty.
\end{aligned}$$

If g is nondecreasing, we may assume that $g(x) \ge 0$ (since $\int_a^b g(x)\,dx = \infty$); then

$$\begin{aligned}\int_a^{x_{r+1}} |f(x)g(x)|\,dx &\ge \sum_{j=0}^{r} \int_{x_j}^{x_{j+1}} |f(x)|g(x)\,dx \ge \rho \sum_{j=1}^{r} g(x_j)\\ &\ge \rho \sum_{j=1}^{r} \frac{\int_{x_{j-1}}^{x_j} g(x)\,dx}{x_j - x_{j-1}}\\ &\ge \frac{\rho}{M} \int_{x_0}^{x_r} g(x)\,dx \to \infty \qquad \text{as} \qquad r \to \infty.\end{aligned}$$

(b) If g is monotonic, then g' does not change sign; hence, $\int_a^c |g'(x)|\,dx = |\int_a^c g'(x)\,dx| = |g(c) - g(a)| \to |g(a)|$ as $c \to b-$.

3.4.18. **(a)** $p \ge 1$: Since $|x^{-p}\cos x| < x^{-p}$, absolutely convergent (Example 3.4.3 and Theorem 3.4.6). $0 < p \le 1$: convergent by Theorem 3.4.10 with $f(x) = \cos x$ and $g(x) = x^{-p}$, but not absolutely convergent because of Theorem 3.4.12; hence, conditionally convergent. $p = 0$: $\int_0^c \cos x\,dx = \sin c$; divergent. $p > 0$: divergent by Theorem 3.4.11).

(b) $p > 1$: absolutely convergent by Theorem 3.4.6, since

$$\left|\frac{\sin x}{x(\log x)^p}\right| \le \frac{1}{x(\log x)^p}, \qquad x > 2,$$

and

$$\int_2^c \frac{dx}{x(\log x)^p}\,dx = \frac{(\log 2)^{1-p} - (\log c)^{1-p}}{p-1} \to \frac{(\log 2)^{1-p}}{p-1} \text{ as } c \to \infty.$$

$p \le 1$: convergent by Theorem 3.4.10 with $g(x) = 1/x(\log x)^p$. However, since $\int_2^\infty g(x)\,dx = \infty$, not absolutely convergent (Theorem 3.4.12); hence, conditionally convergent.

(c) $p > 1$: absolutely convergent by Example 3.4.3 and Theorem 3.4.6, since $|\sin x/x^p \log x| < x^{-p}$. $0 \le p \le 1$: convergent by Theorem 3.4.10; not absolutely convergent, by Theorem 3.4.12; hence, conditionally convergent. $p < 0$: divergent by Theorem 3.4.11.

(d) If $f(x) = x^{-p}\sin 1/x$ and $g(x) = x^{-p-1}$, then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$, so the integral converges absolutely if and only if $p > 0$ (Theorem 3.4.7). Since $f(x) > 0$ for all x in $[1, \infty)$ there is no possibility of conditional conveergence.

(e) Let $f(x) = \dfrac{\sin^2 x \sin 2x}{x^p}$ Write the integral as $I = I_1 + I_2$, where $I_1 = \int_0^1 f(x)\,dx$ and $I_2 = \int_1^\infty f(x)\,dx$ (both improper).e

First consider I_1. Note that $f(x) \geq 0$, $0 \leq x \leq 1$. Since $\lim_{x\to 0+} f(x)/x^{p-3} = 2$ and $\int_0^1 x^{-p+3}\,dx < \infty$ if and only $p < 4$, Theorem 3.4.7 implies that I_1 converges if and only $p < 4$, and the convergence is absolute.

Now consider I_2. Since $|\sin^2 x \sin 2x| \leq 1$, Example 3.4.3 and Theorem 3.4.6 imply that I_1 converges absolutely if $p > 1$. Since $\int_0^x \sin^2 t \sin 2t\,dt = 2\int_0^x \sin^3 t \cos t\,dt = \frac{1}{2}\sin^4 x$ is bounded, Theorem 3.4.10 with $g(x) = x^{-p}$ implies that I_2 converges if $p > 0$; however, Theorem 3.4.12 implies that the convergence is not absolute in this case, since $\int_{j\pi}^{(j+1/2)\pi} \sin^3 x \cos x\,dx = \int_{(j+1/2)\pi}^{(j+1)\pi} \sin^3 x \cos x\,dx = \frac{1}{4}$, $j = 1, 2,3, \dots$, and $\int_0^\infty x^{-p}\,dx = \infty$ if $p > 0$.

Hence, we conclude that I converges absolutely if $1 < p < 4$, or conditionally if $0 < p \leq 1$.

(f) Let $f(x) = \dfrac{\sin x}{(1+x^2)^p}$. Write the integral as $I = I_1+I_2+I_3$, with $I_1 = \int_{-\infty}^{-1} f(x)\,dx$, $I_2 = \int_{-1}^{1} f(x)\,dx$, and $I_3 = \int_1^\infty f(x)\,dx$. Since $|f(x)| < x^{-2p}$, Example 3.4.3 and Theorem 3.4.6 imply that I_3 converges absolutely if $p > 1/2$. Theorem 3.4.10 implies that I_3 converges if $p > 0$, while Theorem 3.4.12 implies that the convergence is not absolute if $0 < p \leq 1/2$; hence, the convergence is conditional in this case. A similar analysis show that I_1 converges absolutely if $p > 1/2$, conditionally if $0 < p \leq 1/2$. Therefore, I converges absolutely if $p > 1/2$, conditionally if $0 < p \leq 1/2$.

3.4.19. $\int_0^c g(x)\sin x\,dx = -g(x)\cos x\big|_0^c + \int_0^c g'(x)\cos x\,dx$. Since $\int_0^\infty g'(x)\cos x\,dx$ converges by Theorem 3.4.10, $\int_0^\infty g(x)\sin x\,dx$ converges if and only if $\lim_{c\to\infty} g(c)\cos c$ exists, which occurs if and only if $L = 0$.

3.4.20.**(a)** $|e^{-sx}h(x)| < |e^{-s_0x}h(x)|$ if $x > 0$ and $s > s_0$. Use Theorem 3.4.6.

(b) Use Theorem 3.4.10, with $f(x) = h(x)e^{-s_0x}$ and $g(x) = e^{-(s-s_0)x}$.

3.4.21. Assume that $\alpha > 1$. If $f(t) \equiv A$, then $\int_0^x t^\alpha f(t)\,dt = \dfrac{Ax^{\alpha+1}}{\alpha+1}$, so we conjecture that $\lim_{x\to\infty} x^{-\alpha-1}\int_0^x t^\alpha f(t)\,dt = \dfrac{A}{\alpha+1}$. To verify this, define

$$F(x) = x^{-\alpha-1}\int_0^x t^\alpha f(t)\,dt - \frac{A}{\alpha+1} = x^{-\alpha-1}\int_0^x t^\alpha (f(t)-A)\,dt.$$

Given $\epsilon > 0$, choose x_0 so that $|f(x) - A| < \epsilon$ if $x \geq x_0$. Then, if $x > x_0$,

$$|F(x)| \leq x^{-\alpha-1}\left(\int_0^{x_0} t^\alpha |f(t)-A|\,dt + \epsilon\int_{x_0}^x t^\alpha\,dt\right) \leq x^{-\alpha-1}\int_0^{x_0} t^\alpha |f(t)-A|\,dt + \epsilon.$$

Since the last integral is a constant indpendent of x, this implies that $\overline{\lim_{x\to\infty}} |F(x)| \le \epsilon$. Since ϵ can be chosen arbitrarly small, $\lim_{x\to\infty} F(x) = 0$.

3.4.22. (A) $\int_a^x f(t)g(t)\,dt = F(x)g(x) - \int_a^x F(t)g'(t)\,dt$. If $|F(x)| \le M$ on $[a,b)$, then (A) implies that

$$\left|\int_a^x f(t)g(t)\,dt\right| \le M\left[g(x) + \int_a^x g'(t)\,dt\right] = M\left(2g(x) - g(a)\right).$$

This yields the conclusion.

3.4.23. Let $\epsilon > 0$ be given. From Exercise 3.4.6, there is a number x_0 in $[a,b)$ such that $F(x) = \int_{x_0}^x f(t)\,dt$ satisfies $|F(x)| < \epsilon$ if $x_0 \le x < b$. Now suppose that $x_0 \le x \le b$; then

$$\int_a^x f(t)g(t)\,dt = \int_a^{x_0} f(t)g(t)\,dt - F(t)g(t)\Big|_{x_0}^x + \int_{x_0}^x F(t)g'(t)\,dt. \qquad \text{(A)}$$

Since $g'(x) \ge 0$, $\left|\int_{x_0}^x F(t)g'(t)\,dt\right| \le \epsilon g(x)$, so (A) implies that $\frac{1}{g(x)}\left|\int_a^x f(t)g(t)\,dt\right| \le \frac{1}{g(x)}\left|\int_a^{x_0} f(t)g(t)\,dt\right| + 3\epsilon$. Since the integral on the right is a constant independent of x and $g(x) \to \infty$ as $x \to b-$, $\overline{\lim_{x\to\infty}} \frac{1}{g(x)}\left|\int_a^x f(t)g(t)\,dt\right| \le 3\epsilon$. Since ϵ can be chosen arbitrarily small, $\lim_{x\to\infty} \frac{1}{g(x)}\left|\int_a^x f(t)g(t)\,dt\right| = 0$

3.4.24. If $a < x < x_1 < b$, then

$$\int_x^{x_1} f(t)g(t)\,dt = -F(t)g(t)\Big|_x^{x_1} + \int_x^{x_1} F(t)g'(t)\,dt.$$

Let $\widehat{F}(x) = \sup_{x\le t<b} |F(t)|$; since $\lim_{t\to b-} F(t) = 0$ (Exercise 3.4.6), $\widehat{F}$ is defined and nonincreasing on $[a,b)$, and (B) $\lim_{x\to b-} \widehat{F}(x) = 0$. Since $|F(t)g'(t)| \le -|\widehat{F}(x)|g'(t)$ $(x \le t < b)$, $\int_x^b |F(t)g'(t)|\,dt < \infty$. Since $\lim_{x_1\to b-} F(x_1)g(x_1) = 0$, letting $x_1 \to b-$ in (A) yields $\int_x^b f(t)g(t)\,dt = F(x)g(x) + \int_x^b F(t)g'(t)\,dt$. This and the assumptions on g imply that $\left|\int_x^b f(t)g(t)\,dt\right| \le 2\widehat{F}(x)g(x)$, and (B) now yields the result.

3.4.25. Use Theorem 3.4.13.

(a) $x = 1/t$; $\int_0^1 x^p \sin 1/x\,dx = \int_1^\infty t^{-p-2}\sin t\,dt$; absolutely convergent if $p > -1$ by Example 3.4.3 and Theorem 3.4.6; conditionally convergent if $-2 < p \le 1$ by Theorems 3.4.10 and 3.4.12; divergent if $p \le -2$ by Theorem 3.4.11.

(b) $x = e^t$, so $t = \log x$; $\int_0^1 |\log x|^p\,dx = \int_{-\infty}^0 t^p e^t\,dt = I_1 + I_2$, where $I_1 = \int_{-\infty}^{-1} t^p e^t\,dt$, $I_2 = \int_{-1}^0 t^p e^t\,dt$. I_1 converges absolutely for every p. Since $\int_{-1}^0 t^p\,dt$ converges if and only if $p > -1$, Theorem 3.4.7 with $f(t) = t^p$ and $g(t) = e^t t^p$ implies that I_2 converges if and only if $p > -1$, since $\lim_{t\to 0-} e^t = 1$. Therefore, I converges absolutely if and only if $p > -1$. There is no possibility of conditional convergence.

(c) $t = \log x$, so $x = e^t$; $I = \int_1^\infty x^p \cos(\log x)\,dx = \int_0^\infty e^{(p+1)t}\cos t\,dt$. Absolutely convergent by Theorem 3.4.6 if $p < -1$, since $\int_0^\infty e^{(p+1)t}\,dt$ converges if $p < -1$. Divergent if $p \ge -1$, by Theorem 3.4.11.

(d) $t = \log x$, so $x = e^t$; $\int_1^\infty (\log x)^p\,dx = \int_0^\infty t^p e^t\,dt$. Since $\lim_{t\to\infty} t^p e^t = \infty$ for all p, Theorem 3.4.7(**b**) with $f(t) = t^p e^t$ and $g(t) = 1$ implies divergence for all p.

(e) $t = x^p$, so $x = t^{1/p}$. If $p > 0$, then $\int_0^\infty \sin x^p\,dx = \frac{1}{p}\int_0^\infty t^{-1+1/p}\sin t\,dt$, which is conditionally convergent if and only if $p > 1$, by Theorems 3.4.10, 3.4.12, and 3.4.11. If $p < 0$, then $I = -\frac{1}{p}\int_0^\infty t^{-1+1/p}\sin t\,dt = (I_1+I_2)/p$, where $I_1 = \int_0^1 t^{-1+1/p}\sin t\,dt$ and $I_2 = \int_1^\infty t^{-1+1/p}\sin t\,dt$. I_2 converges absolutely for all $p < 0$, while I_1 is converges absolutely if $p < -1$. Hence, I converges absolutely if $p < -1$.

3.4.26. **(a)** If $u_2 = u_1\int_x^\infty \frac{dt}{u_1^2}$, then $u_2' = u_1'\int_x^\infty \frac{dt}{u_1^2} - \frac{1}{u_1}$ and $u_2'' = u_1''\int_x^\infty \frac{dt}{u_1^2} - \frac{u_1'}{u_1^2} + \frac{u_1'}{u_1^2} = u_1''\int_x^\infty \frac{dt}{u_1^2}$, so $u_2'' + pu_2 = (u_1'' + pu_1)\int_x^\infty \frac{dt}{u_1^2} = 0$.

If $u_2 = u_1\int_0^x \frac{dt}{u_1^2}$, then $u_2' = u_1'\int_0^x \frac{dt}{u_1^2} + \frac{1}{u_1}$ and $u_2'' = u_1''\int_0^x \frac{dt}{u_1^2} + \frac{u_1'}{u_1^2} - \frac{u_1'}{u_1^2} = u_1''\int_0^x \frac{dt}{u_1^2}$, so $u_2'' + pu_2 = (u_1'' + pu_1)\int_0^x \frac{dt}{u_1^2} = 0$.

(b) Let u be a positive solution of (A) on $[0,\infty)$. If $\int_0^\infty \frac{dx}{u^2(x)} < \infty$, let $y_1 = u\int_x^\infty \frac{dt}{u^2}$ and $y_2 = u$; then

$$y_1y_2' - y_1'y_2 = \left(u\int_x^\infty \frac{dt}{u^2}\right)u' - \left(u'\int_x^\infty \frac{dt}{u^2} - \frac{1}{u}\right)u = 1$$

and $\lim_{x\to\infty}\frac{y_1(x)}{y_2(x)} = \lim_{x\to\infty}\int_x^\infty \frac{dt}{u^2(t)} = 0$. If $\int_0^\infty \frac{dx}{u^2(x)} = \infty$, take $y_1 = u$ and $y_2 = u\int_0^x \frac{dt}{u^2}$; then

$$y_1y_2' - y_1'y_2 = u\left(u'\int_0^x \frac{dt}{u^2} + \frac{1}{u}\right) - u'u\int_0^x \frac{dt}{u^2} = 1$$

and $\lim_{x\to\infty} \dfrac{y_1(x)}{y_2(x)} = \lim_{x\to\infty} \left(\int_0^x \dfrac{dt}{u^2(t)}\right)^{-1} = 0$. In either case, $\left(\dfrac{y_1}{y_2}\right)' = -\dfrac{y_1y_2' - y_1'y_2}{y_2^2} = -\dfrac{1}{y_2^2}$.

3.4.27. **(b)** $u' = -c_1e^{-x} + c_2e^x + \int_0^x h(t)\cosh(x-t)\,dt$; $u'' = c_1e^{-x} + c_2e^x + \int_0^x h(t)\sinh(x-t)\,dt + h(t) = u + h(x)$.

(b) Since $\sinh(x-t) = \dfrac{e^xe^{-t} - e^{-x}e^t}{2}$,

$$u(x) = \left(c_1 - \frac{1}{2}\int_0^x h(t)e^t\,dt\right)e^{-x} + \left(c_2 + \frac{1}{2}\int_0^x h(t)e^{-t}\,dt\right)e^x,$$

so $a(x) = c_1 - \dfrac{1}{2}\int_0^x h(t)e^t\,dt$ and $b(x) = c_2 + \dfrac{1}{2}\int_0^x h(t)e^{-t}\,dt$.

(c) If $\lim_{x\to\infty} a(x) = A$ (finite), then $\int_0^\infty h(t)e^t\,dt$ converges, and $A = c_1 - \dfrac{1}{2}\int_0^\infty h(t)e^t\,dt$. From Exercise 3.4.24 with $f(x) = h(x)e^x$ and $g(x) = e^{-2x}$, $\int_0^\infty h(t)e^{-t}\,dt$ converges, and $\lim_{x\to\infty} e^{2x}\int_x^\infty h(t)e^{-t}\,dt = 0$. Let $B = c_2 + \dfrac{1}{2}\int_0^\infty h(t)e^{-t}\,dt$. Then $e^{2x}[b(x) - B] = -e^{2x}\int_x^\infty h(t)e^{-t}\,dt \to 0$ as $x \to \infty$, and $e^x[u(x) - Ae^{-x} - Be^x] = [a(x) - A] + e^{2x}[b(x) - B] \to 0$ as $x \to \infty$.

(d) Since $\lim_{x\to\infty} b(x)$ =B (finite), $\int_0^\infty f(x)e^{-x}\,dx$ converges. From Exercise 3.4.23 with $f(x) = h(x)e^{-x}$ and $g(x) = e^{2x}$, $\lim_{x\to\infty} e^{-2x}\int_0^x f(x)e^x\,dt = 0$, which implies that $\lim_{x\to\infty} a(x)e^{-2x} = 0$. Now **(a)** implies the conclusion.

3.4.28.

$$u' = c_1y_1' + c_2y_2' + \int_0^x h(t)[y_1(t)y_2'(x) - y_1'(x)y_2(t)]\,dt;$$

$$\begin{aligned} u'' = c_1y_1'' + c_2y_2'' &+ \int_0^x h(t)[y_1(t)y_2''(x) - y_1''(x)y_2(t)]\,dt \\ &+ h(x)[y_1(x)y_2'(x) - y_1'(x)y_2(x)] = -p(x)u + h(x). \end{aligned}$$

$a(x) = c_1 - \int_0^x h(t)y_2(t)\,dt$, $b(x) = c_2 + \int_0^x h(t)y_1(t)\,dt$.

(b) From Exercise 3.4.24 with $f = hy_2$ and $g = y_2/y_2$, $\int_0^\infty h(t)y_1(t)\,dt$ converges and $\lim_{x\to\infty} \dfrac{y_2(x)}{y_1(x)}\int_x^\infty h(t)y_1(t)\,dt = 0$. Let $A = c_1 - \int_0^\infty h(t)y_2(t)\,dt$, $B = c_2 +$

$\int_0^\infty h(t)y_1(t)\,dt$. Then

$$\frac{u(x)-Ay_1(x)-By_2(x)}{y_1(x)} = \int_x^\infty h(t)y_2(t)\,dt - \frac{y_2(x)}{y_1(x)}\int_x^\infty h(t)y_1(t)\,dt \to 0 \text{ as } x\to\infty.$$

(c) From Exercise 3.4.23 with $f = hy_1$ and $g = y_2/y_1$, $\lim_{x\to\infty} \frac{y_1(t)}{y_2(t)}\int_0^x h(t)y_2(t)\,dt = 0$. Therefore, $\lim_{x\to\infty} a(x)\frac{y_1(x)}{y_2(x)} = 0$, so $\lim_{x\to\infty}\frac{u(x)}{y_2(x)} = c_2 + \int_0^\infty h(t)y_1(t)\,dt$.

3.4.29. Let $F(x) = \int_a^x f(t)g(t)\,dt$; then

$$\int_a^x f_1(t)g(t)\,dt = \int_a^x \left(\frac{f_1(t)}{f(t)}\right)F'(t)\,dt = \frac{f_1(t)}{f(t)}F(t)\bigg|_a^x - \int_a^x \left(\frac{f_1(t)}{f(t)}\right)' F(t)\,dt. \quad \text{(A)}$$

Since $\int_a^b f(t)g(t)\,dt$ converges, $\lim_{x\to b-} F(x)$ exists (finite). Since $\left(\frac{f_1}{f}\right)'$ is integrable in $[a,b)$, $\lim_{x\to\infty}\frac{f_1(t)}{f(t)}$ exists (finite). Since $\left(\frac{f_1}{f}\right)'$ is absolutely integrable and F is bounded on $[a,b)$, Theorem 3.4.6 implies that the integral on the right side of (A) converges as $x\to\infty$. This implies the conclusion.

3.4.30. The assumptions imply that for some c in $[x_1,x_2]$,

$$\begin{aligned}\left|\int_{x_1}^{x_2} f(x)g(x)\,dx\right| &= \left|f(c)\int_{x_1}^{x_2} g(x)\,dx\right| \quad \text{(Theorem 3.3.7)}\\ &\ge \rho\int_{x_1}^{x_2}|g(x)|\,dx \ge \rho M.\end{aligned}$$

Therefore, the inequality of Exercise 3.4.7 with f replaced by fgcannot hold if $\epsilon < \rho M$. Hence, $\int_a^b f(x)g(x)\,dx$ diverges.

3.5 ANOTHER LOOK AT THE EXISTENCE OF THE PROPER RIEMANN INTEGRAL

3.5.1. Let $M = \sup_{a\le x\le b} f(x)$, $m = \inf_{a\le x\le b} f(x)$, and $\rho = \sup_{x,x'\in[a,b]} |f(x)-f(x')|$. Then (A) $\rho \le M-m$. If $\epsilon>0$, then $f(x) > M-\epsilon/2$ and $f(x') < m+\epsilon/2$ for some x, x' in $[a,b]$; hence, $f(x)-f(x') \ge M-m-\epsilon$, so $\rho \ge M-m-\epsilon$. Since this is true for every $\epsilon>0$, $\rho\ge M-m$. This and (A) imply that $\rho = M-m$.

3.5.2. If S and S_1 are the sets of discontinuities of f and $|f|$, then (A) $S_1\subset S$. If f is integrable, then S is of Lebesgue measure zero (Theorem 3.5.1, and (A) implies that S_1 is also; hence, $|f|$ is integrable (Theorem 3.5.6). Since $1/f$ is bounded and has the same discontinuities as f if $|f(x)|\ge\rho>0$, the same argument yields the second result.

3.5.3. If $\epsilon > 0$, there are open intervals $\{I_j\}$ and $\{I'_j\}$ such that $S_1 \subset \bigcup_j I_j$ and $S_2 \subset \bigcup_j I'_j$, with $\sum_{j=1}^n L(I_j) < \epsilon/2$ and $\sum_{j=1}^n L(I'_j) < \epsilon/2$, $n \ge 1$. Let $\{I''_1, I''_2, I''_3, I''_4, \dots\} = \{I_1, I'_1, I_2, I'_2, \dots\}$. Then $\sum_{j=1}^n L(I''_j) < \epsilon$, $n \ge 1$. and $S_1 \cup S_2 \subset \bigcup_j I''_j$. Hence, $S_1 \cup S_2$ is of Lebesgue measure zero.

3.5.4. Let S_f and S_g be the sets of discontinuities of f and g, respectively. Since f and g are integrable on $[a, b]$, S_1 and S_2 are of measure zero (Theorem 3.5.6). Hence, $S_1 \cup S_2$ is of measure zero (Exercise 3.5.3). Therefore, the sets of discontinuities of $f + g$ and fg are of measure zero, so $f + g$ and fg are integrable on $[a, b]$ (Theorem 3.5.6).

3.5.5. Let S_f and S_h be the sets of discontinuities of f and h. Since f is integrable on $[a, b]$, S_f is of Lebesgue measure zero (Theorem 3.5.6). Since g is continuous on $[\alpha, \beta]$, $S_h \subset S_f$. (Theorem 2.2.7). Hence, S_h is of Lebesgue measure zero, so h is integrable on $[a, b]$ (Theorem 3.5.6).

3.5.6. Let $\tau_n = \frac{1}{n}\sum_{j=1}^n |F(u_{jn}) - F(v_{jn})|$, $h = F \circ g/(b-a)$, and $P_n = \{x_{0n}, x_{1n}, \dots, x_{nn}\}$, where $x_{jn} = a + j(b-a)/n$, $0 \le j \le n$. Define $M_{jn} = \sup\left\{h(x) \,\middle|\, x_{j-1,n} \le x \le x_{jn}\right\}$ and $m_{jn} = \inf\left\{h(x) \,\middle|\, x_{j-1,n} \le x \le x_{jn}\right\}$. Let $s(P_n)$ and $S(P_n)$ be the lower and upper sums of h over P_n; then (A) $S(P_n) - s(P_n) = \frac{1}{n}\sum_{j=1}^n (M_{jn} - m_{jn})$. Since h is integrable over $[a, b]$ (Exercise 3.5.5), (B) $\lim_{n\to\infty}(S(P_n) - s(P_n)) = 0$ (Exercise 3.2.4). Since $|G(f(u_{jn})) - G(f(v_{jn}))| \le (b-a)(M_{jn} - m_{jn})$ (A) and (B) imply that $\lim_{n\to\infty} \tau_n = 0$.

3.5.7. Let $S = \{x \in [a, b] \mid f(x) \ne 0\}$. $P = \{x_0, x_1, \dots, x_n\}$ be an arbitrary partition of $[a, b]$. None of the intervals $[x_{j-1}, x_j]$ is contained in S, since any subset of S is necessarily of measure zero. Therefore, for each $j \ge 1$, there is a c_j in $[x_{j-1}, x_j]$ such that $f(c_j) = 0$, so $\sum_{j=1}^n f(c_j)(x_j - x_{j-1}) = 0$. Thus, associated with every partition of $[a, b]$ is a Riemann sum of f that equals zero. Hence, $\int_a^b f(x)\,dx = 0$, by Definition 3.1.1.

3.5.8. Apply Exercise 3.5.7 to $h = f - g$. Since $f(x) - g(x) = 0$ except on a set of measure zero, $\int_a^b (f(x) - g(x))\,dx = 0$. Therefore, $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

CHAPTER 4
Infinite Sequences and Series

4.1 SEQUENCES OF REAL NUMBERS

4.1.1. If $\epsilon > 0$, there is an integer N such that $|s_N - s| < \epsilon$. Therefore, $s > s_N - \epsilon \ge -\epsilon$, since $s_N \ge 0$; that is, s is greater than any negative number, and is therefore nonnegative.

4.1.2. **(a)** From Definition 4.1.1, $\lim_{n\to\infty} s_n = s$ (finite) if, for each $\epsilon > 0$, there is an inteer N such that (A) $|s_n - s| < \epsilon$ if $n \ge N$. Now let $t_n = |s_n - s|$. From Definition 4.1.1 with $\{s_n\}$ replaced by $\{t_n\}$ and s replaced by zero, $\lim_{n\to\infty} t_n = 0$ if, for each $\epsilon > 0$, there is an integer N such that (B) $|t_n| < \epsilon$ if $n \ge N$. Since the inequalities (A) and (B) are equivalent, the conclusion follows.

(b) Suppose that $|s_n - s| < t_n$ for $n \ge N_0$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} t_n = 0$ there is an integer N_1 such that $|t_n| < \epsilon$ if $n \ge N_1$. Therefore, $|s - s_n| < \epsilon$ if $n \ge N = \max(N_0, N_1)$. Hence, $\lim_{n\to\infty} s_n = s$, by Definition 4.1.1.

4.1.3.

(a) $|s_n - 2| = \dfrac{1}{n+1} < \epsilon$ if $n \ge N$, where $N > 1/\epsilon$. Therefore, $\lim_{n\to\infty} s_n = 2$.

(b) (A) $|s_n - 1| = \left|\dfrac{\alpha + n}{\beta + n} - 1\right| = \dfrac{|\alpha - \beta|}{|\beta + n|} = \dfrac{|\alpha - \beta|}{n|1 + \beta/n|}$. If $\alpha = \beta$, then $s_n = 1$ for all n, so $\lim_{n\to\infty} s_n = 1$. Now suppose that $\alpha \ne \beta$. First choose $N_1 > 2|\beta|$. If $n \ge N_1$, then $n \ge 2\beta$, so $|\beta/n| < 1/2$, which implies that $|1 + \beta/n| \ge |1 - |\beta/n|| > 1/2$ and $\dfrac{1}{|1 + \beta/n|} < 2$; therefore, (A) implies that (B) $|s_n - 1| < \dfrac{2|\alpha - \beta|}{n}$ if $n \ge N_1$. Now let N_2 be an integer such that $N_2 > 2|\alpha - \beta|/\epsilon$ and let $N = \max(N_1, N_2)$. Then (B) implies that $|s_n - 1| < \epsilon$ if $n \ge N$. Hence, $\lim_{n\to\infty} s_n = 1$.

(c) $|s_n| = \left|\dfrac{1}{n} \sin \dfrac{n\pi}{4}\right| \le \dfrac{1}{n}$. Choose $N > 1/\epsilon$; then $|s_n| < \dfrac{1}{N} < \epsilon$ if $n \ge N$. Hence, $\lim_{n\to\infty} s_n = 0$.

4.1.4. **(a)** $\lim_{n\to\infty} s_1 = 1/2$:

$$\begin{aligned}\left|s_n - \frac{1}{2}\right| &= \left|\frac{n}{2n+\sqrt{n+1}} - \frac{1}{2}\right| \\ &= \left|\frac{2n-2n-\sqrt{n+1}}{2(2n+\sqrt{n+1})}\right| = \frac{\sqrt{n+1}}{2(2n+\sqrt{n+1})} \\ &\le \frac{\sqrt{n+1}}{4(n+1)} = \frac{1}{4\sqrt{n+1}}\end{aligned}$$

if $n > 3$. If $\epsilon > 0$ is given then choose $N > 3$ so that $4\sqrt{n+1} > 1/\epsilon$. Then $|s_n - 1/2| < \epsilon$ if $n \ge N$.

(b) $\lim_{n\to\infty} s_1 = 1/2$:

$$\frac{n^2+2n+2}{n^2+n} = \frac{(n^2+n)+(n+1)}{n^2+n} = 1 + \frac{1}{n},$$

so

$$|s_n - 1| = \frac{1}{n}.$$

If $\epsilon > 0$ is given, choose N so that $N > 1/\epsilon$. Then $|s_n - 1| < \epsilon$ if $n \ge N$.

(c) $\lim_{n\to\infty} s_n = 0$; since $|\sin n| < 1$,

$$|s_n - 0| = \frac{|\sin n|}{\sqrt{n}} = \frac{1}{\sqrt{n}}.$$

If $\epsilon > 0$ is given, choose N so that $N > 1/\epsilon^2$. Then $|s_n| < \epsilon$ if $n \ge N$.

(d)

$$\begin{aligned}s_n &= \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} \\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}.\end{aligned}$$

Note that $0 < s_n < 1/2$. We'll show that $\lim_{n\to\infty} s_n = 1/2$.

$$\begin{aligned}s_n - \frac{1}{2} &= \frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2} \\ &= \frac{2n-\sqrt{n^2+n}-n}{2(\sqrt{n^2+n}+n)} = -\frac{s_n}{2(\sqrt{n^2+n}+n)}.\end{aligned}$$

Therefore,

$$\left|s_n - \frac{1}{2}\right| = \frac{|s_n|}{2(\sqrt{n^2+n}+n)} < \frac{1}{8n}.$$

Given $\epsilon > 0$ choose $N \ge 1/8\epsilon$; then

$$\left| s_n - \frac{1}{2} \right| < \epsilon \quad \text{if} \quad n > N.$$

4.1.5. The integer N in Definition 4.1.1 and does not depend upon ϵ if and only if the sequence is constant for sufficiently large n. To see that this is sufficent, suppose that that $s_n = s$ for all $n \ge N_1$. Then $|s_n - s| = 0$ if $n \ge N_1$. Therefore, if ϵ is an arbitrary positive number, then $|s_n - s| < \epsilon$ if $n \ge N_1$. To see that it is necessary, suppose that there is an integer N such that $|s_n - s| < \epsilon$ if $n \ge N$, for every $\epsilon > 0$. Fixing $n \ge N$ and letting $\epsilon \to 0+$, we see that $s_n = s$.

4.1.6. Let $\epsilon > 0$. Since $\lim_{n\to\infty} s_n = s$ there is an integer N such that $|s_n - s| < \epsilon$ if $n \ge N$. since $||s_n| - |s|| \le |s_n - s|$ for all n, $||s_n| - |s|| < \epsilon$ for $n \ge N$; hence, $\lim_{n\to\infty} |s_n| = |s|$.

4.1.7. Let $\epsilon > 0$. Since $\lim\limits_{n\to\infty} s_n = s$ there is an integer N_1 such that (A) $|s_n - s| < \epsilon$ if $n \ge N_1$. By assumption, there is an integer N_2 such that (B) $|t_n - s_n| < \epsilon$ if $n \ge N_2$. Both (A) and (B) hold if $n \ge N = \max(N_1, N_2)$, so $|t_n - s| \le |t_n - s_n| + |s_n - s| < 2\epsilon$ if $n \ge N$. Hence, $\lim_{n\to\infty} t_n = s$, by Definition 4.1.1.

4.1.8. Let $\alpha = \inf\{s_n\}$. If $\alpha > -\infty$, Theorem 1.1.8 implies that if $\epsilon > 0$ then $\alpha \le s_N < \alpha + \epsilon$ for some integer N. Since $\alpha \le s_n \le s_N$ if $n \ge N$, it follows that $\alpha \le s_n < \alpha + \epsilon$ if $n \ge N$. This implies that $|s_n - \alpha| < \epsilon$ if $n \ge N$, so $\lim\limits_{n\to\infty} s_n = \alpha$ by Definition 4.1.1. If $\alpha = -\infty$ and b is any real number, then $s_N < b$ for some integer N. Then $s_n < b$ for $n \ge N$, so $\lim\limits_{n\to\infty} s_n = -\infty$.

4.1.9. **(a)**
$$\begin{aligned} s_{n+1} - s_n &= \frac{\alpha + n + 1}{\beta + n + 1} - \frac{\alpha + n}{\beta + n} \\ &= \frac{(\alpha + n + 1)(\beta + n) - (\alpha + n)(\beta + n + 1)}{(\beta + n + 1)(\beta + n)} \\ &= \frac{\beta - \alpha}{(\beta + n + 1)(\beta + n)}. \end{aligned}$$

Therefore, $\{s_n\}$ is increasing if $\beta > \alpha$, decreasing if $\alpha < \beta$. Since $|s_n| = \left|\dfrac{\alpha + n}{\beta + n}\right| = \dfrac{|1 + \alpha/n|}{1 + \beta/n} \le 1 + |\alpha|$, $n \ge 1$, $\{s_n\}$ is bounded. Since $\{s_n\}$ is monotonic and bounded, Theorem 4.1.6 implies that $\{s_n\}$ converges.

(b) $\dfrac{s_{n+1}}{s_n} = \dfrac{(n+1)!}{(n+1)^{n+1}} \dfrac{n^n}{n!} = \left(\dfrac{n}{n+1}\right)^n < 1$, so $\{s_n\}$ is decreasing. Since $s_n > 0$ for all n, Theorem 4.1.6 implies that $\{s_n\}$ converges.

(c)

$$s_{n+1} - s_n = \frac{r^{n+1}}{1 + r^{n+1}} - \frac{r^n}{1 + r^n}$$
$$= \frac{r^{n+1}(1 + r^n) - r^n(1 + r^{n+1})}{(1 + r^{n+1})(1 + r^n)}$$
$$= \frac{r^n(r - 1)}{(1 + r^{n+1})(1 + r^n)}.$$

Therefore, $\{s_n\}$ is decreasing if $0 < r < 1$, increasing if $r > 1$. Since $0 < s_n < 1$ for all n, Theorem 4.1.6 implies that $\{s_n\}$ converges.

(d) $\dfrac{s_{n+1}}{s_n} = \dfrac{(2n + 2)!}{2^{2n+2}[(n + 1)!]^2} \dfrac{2^{2n}(n!)^2}{(2n)!} = \dfrac{(2n + 2)(2n + 1)}{4(n + 1)^2} = \dfrac{2n + 1}{2n + 2} < 1$, so $\{s_n\}$ is decreasing. Since $s_n > 0$ for all n, Theorem 4.1.6 implies that $\{s_n\}$ converges.

4.1.10. If $x > 0$, then $\mathrm{Tan}^{-1}x > 0$. Moreover $x_{n+1} - x_n = f(x_n)$ where $f(x) = \mathrm{Tan}^{-1}(x) - x$. Since $f(0) = 0$ and $f'(x) = \dfrac{1}{1 + x^2} - 1 < 0$ if $x \ne 0$, e $f(x) < 0$ if $x > 0$. Therefore, if $x_0 > 0$, then $0 < x_1 < x_0$. By induction, $0 < x_{n+1} < x_n$. By Theorem 4.1.6, $\lim_{n\to\infty} x_n = \inf\{s_n\}$.

4.1.11. **(a)** If $n \ge 0$, then $s_{n+1}^2 - A = \dfrac{1}{4}\left(s_n + \dfrac{A}{s_n}\right)^2 - A = \dfrac{1}{4}\left(s_n^2 + 2A + \dfrac{4}{s_n^2}\right) - \dfrac{4A}{4} = \dfrac{1}{4}\left(s_n - \dfrac{A}{s_n}\right)^2 \ge 0$, so $s_{n+1} \ge \sqrt{A}$ if $n \ge 0$.

(b) If $n \ge 1$, then $s_{n+1} - s_n = \dfrac{1}{2}\left(s_n + \dfrac{A}{s_n}\right) - s_n = \dfrac{1}{2}\left(\dfrac{A}{s_n} - s_n\right) = \dfrac{A - s_n^2}{2s_n} \le 0$ from **(a)**.

(c) Since $\{s_n\}$ is nondecreasing and bounded below, it is convergent by Theorem 4.1.6(**b**)

(d) Let $s = \lim_{n\to\infty}$. Then $s = \lim\limits_{n\to\infty} \dfrac{1}{2}\left(s_n + \dfrac{A}{s_n}\right) = \dfrac{1}{2}(s + \dfrac{A}{s})$, which implies that $s^2 = A$; hence, $s = \sqrt{A}$.

4.1.12. If $\{s_n\}$ is nondecreasing, then $\sup\{s_n\} = \infty$, while if $\{s_n\}$ is nonincreasing, then $\inf\{s_n\} = -\infty$. In either case the conclusion follows from Theorem 4.1.6.

4.1.13. Suppose that $s_n = f(n)$ for $n \ge N_1$. Let $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = L$ there is an integer $N > N_1$ such that $|f(x) - L| < \epsilon$ if $x > N$, so $|s_n - L| = |f(n) - L| < \epsilon$ if $n \ge N$. Therefore, $\lim\limits_{n\to\infty} s_n = L$.

4.1.14. **(a)** $f(x) = \dfrac{\alpha + x}{\beta + x} = \dfrac{1 + \alpha/x}{1 + \beta/x} \to 1$ as $t \to \infty$; therefore, $\lim\limits_{n\to\infty} s_n = 1$.

(b) $f(x) = \cos\dfrac{1}{x}$ as $t \to \infty$; therefore, $\lim\limits_{n\to\infty} s_n = 1$.

(c) $f(x) = x\sin\dfrac{1}{x} = \dfrac{\sin 1/x}{1/x}$; by L'Hospital's rule, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{-1/x^2\cos 1/x}{-1/x^2} = \lim\limits_{x\to\infty}\cos\dfrac{1}{x} = 1$; therefore, $\lim_{n\to\infty} s_n = 1$.

(d) $f(x) = \log x - x = -x\left(1 - \dfrac{\log x}{x}\right)$; since $\lim_{x\to\infty} \dfrac{\log x}{x} = \lim_{x\to\infty} \dfrac{1}{x} = 0$, $\lim_{n\to\infty} f(x) = -\infty$, so $\lim_{n\to\infty} s_n = -\infty$.

(e) $f(x) = \log(x+1) - \log(x-1) = \log\left(\dfrac{x+1}{x-1}\right) = \log\left(\dfrac{1+1/x}{1-1/x}\right) \to \log 1 = 0$ as $x \to \infty$; $\lim_{n\to\infty} s_n = 0$.

4.1.15. If $c = 0$, then $cs_n = 0$ for all n, so $\lim_{n\to\infty}(cs_n) = 0 = 0 \cdot es$. Now suppose that $c \ne 0$ and $\epsilon > 0$. Since $\lim_{n\to\infty} s_n = s$, there is an integer N such that $|s - s_n| < \epsilon$ if $n \ge N$; then $|cs_n - cs| \le |c|\epsilon$ if $n \ge N$, so $\lim_{n\to\infty}(cs_n) = cs$.

4.1.16. Suppose that $s = \infty$ and $c > 0$. For arbitrary a there is an integer N such that (A) $s_n > a/c$ if $n \ge N$. Then $cs_n > a$ if $n \ge N$, so $\lim_{n\to\infty}(cs_n) = \infty = cs$.

Suppose that $s = \infty$ and $c < 0$. For arbitrary a there is an integer N such that (A) $s_n > a/c$ if $n \ge N$. Then $cs_n < a$ if $n \ge N$, so $\lim_{n\to\infty}(cs_n) = -\infty = cs$.

Suppose that $s = -\infty$ and $c > 0$. For arbitrary a there is an integer N such that (A) $s_n < a/c$ if $n \ge N$. Then $cs_n < a$ if $n \ge N$, so $\lim_{n\to\infty}(cs_n) = -\infty = cs$.

Suppose that $s = -\infty$ and $c < 0$. For arbitrary a there is an integer N such that (A) $s_n < a/c$ if $n \ge N$. Then $cs_n > a$ if $n \ge N$, so $\lim_{n\to\infty}(cs_n) = \infty = cs$.

4.1.17. If $\epsilon > 0$, there are integers N_1 and N_2 such that (A) $|s_n - s| < \epsilon$ if $n \ge N_1$ and (B) $|t_n - t| < \epsilon$ if $n \ge N_2$. If $N = \max(N_1, N_2)$, then (A) and (B) both hold when $n \ge N$, so $|(s_n + t_n) \pm (s + t)| \le |s_n - s| + |t_n - t| < 2\epsilon$ if $n \ge N = \max(N_1, N_2)$. Therefore, $\lim_{n\to\infty}(s_n \pm t_n) = s \pm t$.

4.1.18. The equality holds if s and t are both finite (Exercise 4.1.17). Now suppose that $s = \infty$ and $t > -\infty$. Then $\alpha = \inf\{t_n\}$ is finite and if a is arbitrary there is an integer N such that $s_n > a - \alpha$ if $n \ge N$. Therefore, $s_n + t_n > a$ if $n \ge N$, so $\lim_{n\to\infty}(s_n + t_n) = \infty = s + t$.

Now suppose that $s = -\infty$ and $t < \infty$. Then $\beta = \sup\{t_n\}$ is finite and if a is arbitrary, there is an integer N such that $s_n < a - \beta$ if $n \ge N$. Therefore, $s_n + t_n < a$ if $n \ge N$, so $\lim_{n\to\infty}(s_n + t_n) = -\infty = s + t$.

4.1.19. There is an integer N such that $|t_n - t| < (1-\rho)|t|$ if $n \ge N$. (Take $\epsilon = (1-\rho)|t|$ in the definition of limit); therefore,

$$t - (1-\rho)|t| < t_n < t + (1-\rho)|t|, \quad n \ge N.$$

If $t > 0$, the first inequality is equivalent to $t_n \ge \rho t$, $n \ge N$; if $t < 0$, the second inequality is equivalent to $t_n \le \rho t$, $n \ge N$.

4.1.20. $s_n = \left(\dfrac{1+t_n}{1-t_n}\right)s$; therefore, $s_n - s = \left(\dfrac{1+t_n}{1-t_n} - 1\right)s = \dfrac{2t_n s}{1-t_n}$. Since $\lim_{n\to\infty} t_n = 0$ there is an integer N_1 such that $|t_n| < 1/2$ if $n \ge N_1$. Then $|1 - t_n| \ge 1 - |t_n| > \dfrac{1}{2}$ and $\dfrac{1}{|1-t_n|} < 2$ if $n \ge N_1$, so $|s_n - s| < 4|st_n|$ if $n \ge N_1$. If $\epsilon > 0$, choose $N \ge N_1$ so that

$|t_n| < \dfrac{\epsilon}{4|s|}$ if $n \ge N$. Then $|s_n - s| < \epsilon$ if $n \ge N$.

4.1.21. We showed in the text that the equality holds if s and t are both finite, so we consider only the case where $s = \pm\infty$ and $t \ne 0$. Our assumptions concerning $\{t_n\}$ for sufficiently large n are justified by Exercise 4.1.19 if $|t| < \infty$, or by the definitions of $\lim_{n\to\infty} t_n = \pm\infty$ if $|t| = \infty$.

Suppose that $s = \infty$, $t > 0$, and $0 < b < t$. There is an integer N_1 such that $s_n > 0$ and $t_n > b$ if $n \ge N_1$, so (A) $s_n t_n > b s_n$ if $n \ge N_1$. For arbitrary a there is an integer $N \ge N_1$ such that $s_n > a/b$, and therefore $b s_n > a$, if $n \ge N$. From (A), $s_n t_n > a$ if $n \ge N_1$, so $\lim_{n\to\infty} s_n t_n = \infty = st$.

Suppose that $s = \infty$, $t < 0$, and $t < b < 0$. There is an integer N_1 such that $s_n > 0$ and $t_n < b$ if $n \ge N_1$, so (B) $s_n t_n < b s_n$ if $n \ge N_1$. For arbitrary a there is an integer $N \ge N_1$ such that $s_n > a/b$, and therefore $b s_n < a$, if $n \ge N$. From (B), $s_n t_n < a$ if $n \ge N_1$, so $\lim_{n\to\infty} s_n t_n = -\infty = st$.

Suppose that $s = -\infty$, $t > 0$, and $0 < b < t$. There is an integer N_1 such that $s_n < 0$ and $t_n > b$ if $n \ge N_1$, so (C) $s_n t_n < b s_n$ if $n \ge N_1$. For arbitrary a there is an integer $N \ge N_1$ such that $s_n < a/b$, and therefore $b s_n < a$, if $n \ge N$. From (C), $s_n t_n < a$ if $n \ge N_1$, so $\lim_{n\to\infty} s_n t_n = -\infty = st$.

Suppose that $s = -\infty$, $t < 0$, and $t < b < 0$. There is an integer N_1 such that $s_n < 0$ and $t_n < b$ if $n \ge N_1$, so (D) $s_n t_n > b s_n$ if $n \ge N_1$. For arbitrary a, there is an integer $N \ge N_1$ such that $s_n < a/b$, and therefore $b s_n > a$, if $n \ge N$. From (D), $s_n t_n > a$ if $n \ge N_1$, so $\lim_{n\to\infty} s_n t_n = \infty = st$.

4.1.22. The case where where s and t are both finite is covered in the text. Therefore, we need only consider the cases where either s or t (but not both) is $\pm\infty$, and $t \ne 0$. Let $\epsilon > 0$.

Suppose that s is finite and $t = \pm\infty$. Since s is finite there is a constant M such that $|s_n| \le M$ for all n (Theorem 4.1.4), and there is an integer N such that $|t_n| > M/\epsilon$ if $n \ge N$, and therefore $|s_n t_n| < \epsilon$, if $n \ge N$. Therefore, $\lim_{n\to\infty} s_n/t_n = 0 = s/t$.

Suppose that $|s| = \infty$ and $0 < |t| < \infty$. Then there is an integer N_1 such that $0 < |t_n| < 2|t|$ and the products $\{s_n t_n\}_{n=N_1}^{\infty}$ have the sign of s/t if $n \ge N_1$. Therefore, s_n/t_n is defined and $|s_n/t_n| > |s_n|/2|t|$ if $n \ge N_1$. If a is an arbitrary real number there is an integer $N \ge N_1$ such that $|s_n| \ge 2|a||t|$, and therefore $|s_n/t_n| > |a|$ if $n \ge N$. Hence, $\lim_{n\to\infty} s_n/t_n = s/t$.

4.1.23. Since $\{s_n\}$ is bounded below there is a number α such that $s_n > \alpha$ for all n. Since $\{s_n\}$ does not diverge to ∞, there is a number β such that $s_n < \beta$ for infinitely many n. If $m_k = \inf\{s_k, s_{k+1}, \dots, s_{k+r}, \dots\}$, then $\alpha \le m_k \le \beta$, so $\{m_k\}$ is bounded. Since $\{m_k\}$ is nondecreasing it converges, by Theorem 4.1.6. Let (A) $\underline{s} = \lim_{k\to\infty} m_k$. If $\epsilon > 0$, then $m_k > \underline{s} - \epsilon$ for large k, and since $s_n \ge m_k$ for $n \ge k$, $s_n > \underline{s} - \epsilon$ for large n. If there were an ϵ such that $s_n < \underline{s} + \epsilon$ for only finitely many n, there would be an integer K such that $s_n \ge \underline{s} + \epsilon$ if $n \ge K$. However, this implies that $m_k \ge \underline{s} + \epsilon$ if $k \ge K$, which contradicts (A). Therefore, $\underline{s}$ has the stated properties.

If $t > \underline{s}$, the inequality $s_n > t-(t-\underline{s})/2 = \underline{s}+(t-\underline{s})/2$ cannot hold for all large n, since $s_n < \underline{s}+(t-\underline{s})/2$ for infinitely many n. If $t < \underline{s}$, the inequality $s_n < t+(\underline{s}-t)/2 = \underline{s}-(\underline{s}-t)/2$ cannot hold for infinitely many n, because $s_n > \underline{s}-(\underline{s}-t)/2$ for large n. Therefore, $\underline{s}$ is the only real number with the stated properties.

4.1.25. **(a)** Since $s_{2m} = 1$ and $s_{2m+1} = -1$, $\overline{s} = 1$ and $\underline{s} = -1$.

(b) Since $s_{2m} = \left(2+\dfrac{3}{2m}\right)$ and $s_{2m+1} = -\left(2+\dfrac{3}{2m+1}\right)$, $\overline{s} = 2$ and $\underline{s} = -2$.

(c) Since $s_{2m} = \dfrac{6m+1}{2m}$ and $s_{2m+1} = -\dfrac{2m+2}{2m+1}$, $\overline{s} = 3$ and $\underline{s} = -1$.

(d) Since the numbers 0, $\sqrt{3}/2$, and $-\sqrt{3}/2$ (and only these) appear infinitely many times in the sequence, $\overline{s} = \sqrt{3}/2$ and $\underline{s} = -\sqrt{3}/2$.

4.1.26. If $\gamma > 0$, then $|s_n| > \gamma/2$ for large n. If s_n is not sign-constant for large n, then $s_n > \gamma/2$ and $s_m < -\gamma/2$ for infinitely many m and n. Therefore, $\underline{s} \le -\gamma/2 \le \gamma/2 \le \overline{s}$, and $\{s_n\}$ diverges (Theorem 4.1.12).

4.1.27. Choose N so that $|s_n - s_m| < \epsilon$ if $n, m \ge N$. Then $|s_n| \le |s_n - s_N| + |s_N| < \epsilon + |s_N|$ if $n \ge N$.

4.1.28. **(a)** If $\underline{s} = \infty$, then $\lim_{n\to\infty} s_n = \infty$ (Definition 4.1.10 and therefore $\lim_{n\to\infty}(-s_n) = -\infty$ (Theorem 4.1.8), so $\overline{\lim_{n\to\infty}}(-s_n) = -\infty$ (Definition 4.1.10). If $\underline{s} < \infty$ and $\epsilon > 0$, then $s_n > \underline{s}-\epsilon$ for large n and $s_n < \underline{s}+\epsilon$ for infinitely many n; hence, $-s_n < -\underline{s}+\epsilon$ for large n and $-s_n > -\underline{s}-\epsilon$ for infinitely many n. The uniqueness of $\overline{\lim_{n\to\infty}}(-s_n)$ (Theorem 4.1.9**(a)**) implies the conclusion.

(b) $\overline{s} = \overline{\lim_{n\to\infty}}[-(-s_n)] = -\underline{\lim_{n\to\infty}}(-s_n)$ (by **(a)**), which yields the conclusion.

4.1.29. **(a)** Obvious if $\overline{s}+\overline{t} = \infty$. If $\overline{s}+\overline{t} = -\infty$, assume without loss of generality that $\overline{s} = -\infty$ and $t_n \le b < \infty$ for all n. If β is arbitrary, then $s_n < \beta - b$, and therefore $s_n + t_n < \beta$, for large n; hence, $\overline{\lim_{n\to\infty}}(s_n+t_n) = -\infty$. Now suppose that $\overline{s}$ and $\overline{t}$ are finite and $\epsilon > 0$. Then $s_n < \overline{s}+\epsilon/2$ and $t_n < \overline{t}+\epsilon/2$, so (A) $s_n + t_n < \overline{s}+\overline{t}+\epsilon$ for large n. Therefore, $\overline{\lim_{n\to\infty}}(s_n+t_n) < \infty$. By definition, (B) $s_n+t_n > -\epsilon + \overline{\lim_{j\to\infty}}(s_j+t_j)$ for infinitely many n. Since (A) and (B) must both hold for some n, $-\epsilon + \overline{\lim_{j\to\infty}}(s_j+t_j) < \overline{s}+\overline{t}+\epsilon$ if $\epsilon > 0$. Letting $\epsilon \to 0+$ yields the result.

(b) Applying **(a)** to $\{-s_n\}$ and $\{-t_n\}$ yields

$$\overline{\lim_{n\to\infty}}[-(s_n+t_n)] \le \overline{\lim_{n\to\infty}}(-s_n) + \overline{\lim_{n\to\infty}}(-t_n),$$

and Exercise 4.1.28**(a)** implies the result.

4.1.30. **(a)**(i). Trivial if $\overline{s}\overline{t} = \infty$. If $\overline{s}\overline{t} < \infty$, then (A) $s_n t_n < (\overline{s}+\epsilon)(\overline{t}+\epsilon)$ for large n, and (B) $s_n t_n > -\epsilon + \overline{\lim_{j\to\infty}} s_j t_j$ for infinitely many n. Since (A) and (B) both hold for some n, $-\epsilon + \overline{\lim_{j\to\infty}} s_j t_j < (\overline{s}+\epsilon)(\overline{t}+\epsilon)$ for small positive ϵ. This implies **(i)**.

(a)(ii). Obvious if $\underline{s}\,\underline{t} = 0$. If $\underline{s} = \infty$, $\underline{t} > \tau > 0$, and $\alpha > 0$;, then $t_n > \tau$, $s_n > \alpha/\tau$, and $s_n t_n > \alpha$ if n is sufficiently large; hence, $\lim_{n\to\infty} s_n t_n = \infty$. If $0 < \epsilon < \underline{s}, \underline{t} < \infty$, then (A)

$s_nt_n > (\underline{s}-\epsilon)(\underline{t}-\epsilon)$ for n sufficiently large, and (B) $s_nt_n < \epsilon + \underline{\lim}_{j\to\infty} s_jt_j$ for infinitely many n. Since (A) and (B) both hold for some n, $(\underline{s}-\epsilon)(\underline{t}-\epsilon) < \epsilon + \underline{\lim}_{j\to\infty} s_jt_j$ for small positive ϵ. This implies (A).

(b) (i)

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\, s_nt_n &= -\underline{\lim}_{n\to\infty}(-s_n)t_n && \text{(Exercise 4.1.28(}\mathbf{a}\text{))}\\
&\le -\underline{t}\,\underline{\lim}_{n\to\infty}(-s_n) && \text{(Exercise 4.1.30(}\mathbf{a}\text{))}\\
&= \ \overline{s}\,\underline{t} && \text{(Exercise 4.1.28(}\mathbf{b}\text{).)}
\end{aligned}$$

(b) (ii)

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\, s_nt_n &= -\overline{\lim_{n\to\infty}}(-s_n)t_n && \text{(Exercise 4.1.28(}\mathbf{b}\text{))}\\
&\le -\overline{t}\,\overline{\lim_{n\to\infty}}(-s_n) && \text{(Exercise 4.1.30(}\mathbf{b}\text{))}\\
&= \ \underline{s}\,\underline{t} && \text{(Exercise 28(}\mathbf{a}\text{))}
\end{aligned}$$

4.1.31. **(a)(i)** Since $s = \overline{s}$ (Theorem 4.1.12), $\overline{\lim_{n\to\infty}}\, s_nt_n \le s\overline{t}$ (Exercise 4.1.30**(a)(i)**); hence, it suffices to show that $\overline{\lim_{n\to\infty}}\, s_nt_n \ge s\overline{t}$. This is obvious if $s\overline{t} = 0$. Suppose that $s\overline{t} = \infty$ and $\beta > 0$. If $0 < \sigma < \overline{t}$ and $s = \infty$, then $t_n > \sigma$ for infinitely many n and $s_n > \beta/\sigma$ for all but finitely many n. Therefore $s_nt_n > \beta$, for infinitely many n, so $\overline{\lim_{n\to\infty}}\, s_nt_n = \infty = s\overline{t}$. A similar argument applies if $\overline{t} = \infty$ and $s > 0$. Now suppose that $0 < s, \overline{t} < \infty$ and $0 < \epsilon < s$, $0 < \epsilon < \overline{t}$. Then $(s-\epsilon)(\overline{t}-\epsilon) < s_nt_n$ for infinitely many n. Since $s_nt_n < \epsilon + \overline{\lim_{j\to\infty}}\, s_jt_j$ for large n, $(s-\epsilon)(\overline{t}-\epsilon) < \epsilon + \overline{\lim_{j\to\infty}}\, s_jt_j$, if ϵ is sufficiently small. This implies the conclusion. **(a)(ii)**. From 26**(a)(ii)**, it suffices to show that (A) $\underline{\lim}_{n\to\infty} s_nt_n \le s\underline{t}$. Obvious if $s\underline{t} = \infty$. If s and $\underline{t}$ are finite (obviously ≥ 0), then $s_nt_n < (s+\epsilon)(\underline{t}+\epsilon)$ for infinitely many n and $s_nt_n > -\epsilon + \underline{\lim}_{j\to\infty} s_jt_j$ for sufficiently large n; hence, $-\epsilon + \underline{\lim}_{j\to\infty} s_jt_j < (s+\epsilon)(\underline{t}+\epsilon)$ if $\epsilon > 0$. This implies (A).

(b) Use Exercises 4.1.28 and 4.1.31**(a)**.

4.1.32. Let $c_1, c_2, \dots, c_k$ be the distinct terms in $\{s_n\}$. If $k = 1$, the conclusion is obvious. If $k > 1$ let $\epsilon_0 = \min\left\{|c_i - c_j| \,\middle|\, i \ne j\right\}$. Since $\{s_n\}$ converges, there is an integer N such that $|s_n - s_m| < \epsilon_0$ if $n, m \ge N$ (Theorem 4.1.13). Therefore, s_n must be constant for $n \ge N$.

4.1.33. If $s_1 = s_0$, then $s_n = s_0$ for all n, soe $\lim_{n\to\infty} s_n = s_0$. Now supppose $s_1 \ne s_0$. $s_{n+1} - s_n = \dfrac{s_n + s_{n-1}}{2} - s_n = \dfrac{s_{n-1} - s_n}{2}$, so $|s_{n+1} - s_n| = \dfrac{|s_n - s_{n-1}|}{2}$. From this and induction, $|s_{n+1} - s_n| = \dfrac{|s_1 - s_0|}{2^n}$, $n \ge 1$. Now, if $n > m$,

$$\begin{aligned}
|s_n - s_m| &= |(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{m+1} - s_m)|\\
&\le |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m|\\
&\le \frac{|s_1 - s_0|}{2^m}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right) < \frac{|s_1 - s_0|}{2^{m-1}},
\end{aligned}$$

since $\sum_{k=0}^{\infty} 2^{-k} = 2$. If $\epsilon > 0$, choose N so that $2^{N-1} > 1/(|s_1 - s_0|\epsilon)$. Then $|s_n - s_m| < \epsilon$ if $n > m > N$.

4.1.34. Suppose that s is finite and $\epsilon > 0$. Choose m so that $|s_n - s| < \epsilon$ if $n > m$. Then, if $n > m$,

$$\begin{aligned} |t_n - s| &= \left| \frac{(s_1 - s) + (s_2 - s) + \cdots + (s_n - s)}{n} \right| \\ &\le \left| \frac{(s_1 - s) + \cdots + (s_m - s)}{n} \right| + \frac{|s_{m+1} - s| + \cdots + |s_n - s|}{n} \\ &\le \frac{|(s_1 - s) + \cdots + (s_m - s)|}{n} + \frac{n-m}{n}\epsilon. \end{aligned}$$

Since the numerator of the first fraction on the left is is independent of n,e $\overline{\lim_{n\to\infty}} |t_n - s| \le \epsilon$. Since ϵ can be made arbitrarily small, $\lim_{n\to\infty} |t_n - s| = 0$; hence, $\lim_{n\to\infty} t_n = s$.

If $s = \infty$ and $b > 0$, choose m so that $s_n > b$ if $n > m$. Then, if $n > m$,

$$t_n \ge \frac{s_{m+1} + \cdots + s_n}{n} - \frac{|s_1 + \cdots + s_m|}{n} > \frac{n-m}{n}b - \frac{|s_1 + \cdots + s_m|}{n}.$$

Since the numerator of the second fraction on the right is independent of n, $\underline{\lim_{n\to\infty}}\ t_n \ge b$. Since b is arbitrary, $\lim_{n\to\infty} t_n = \infty$. If $s = -\infty$, apply this result to $\{-s_n\}$.

4.1.35. **(a)** Suppose that M is an integer $> \alpha$. It suffices to show that

$$\lim_{n\to\infty} \left(1 - \frac{\alpha}{M}\right)\left(1 - \frac{\alpha}{M+1}\right)\cdots\left(1 - \frac{\alpha}{n}\right) = 0.$$

Denote the product here by a_n. Then $\log a_n = \sum_{k=M}^{n} \log\left(1 - \frac{\alpha}{k}\right)$. If $0 < x < 1$, then $\log(1-x) < -x$ (Theorem 2.5.4), so

$$\log a_n < -\alpha \sum_{k=M}^{n} \frac{1}{k} < -\alpha \sum_{k=M}^{n} \int_k^{k+1} \frac{dx}{x} = -\alpha \int_M^{n+1} \frac{dx}{x} = -\alpha \log \frac{n+1}{M};$$

hence, $\lim_{n\to\infty} \log a_n = -\infty$, so $\lim_{n\to\infty} a_n = 0$.

(b) $\left|\binom{q}{n}\right|$ is of the form in **(a)**, with $\alpha = q + 1$.

4.2 EARLIER TOPICS REVISITED WITH SEQUENCES

4.2.1. If $\lim_{k\to\infty} s_{n_k}$ exists, there is an integer N such that

$$\left|s_{n_i} - s_{n_j}\right| = \left|(-1)^{n_i}\left(1 + \frac{1}{n_i}\right) - (-1)^{n_j}\left(1 + \frac{1}{n_j}\right)\right| < 1$$

if $i, j \geq N$. This is impossible unless n_i and n_j have the same parity if $i, j \geq k$.

4.2.4. If $s = \infty$ and a is arbitrary, there is an integer N such that $s_n > a$ if $n \geq N$. There is an integer K such that $n_k \geq N$ if $k \geq K$; therefore, $s_{n_k} > a$ if $k > K$, so $\lim_{k\to\infty} s_{n_k} = \infty$.

If $s = -\infty$ and a is arbitrary, there is an integer N such that $s_n < a$ if $n \geq N$. There is an integer K such that $n_k \geq N$ if $k \geq K$; therefore, $s_{n_k} < a$ if $k > K$, so $\lim_{k\to\infty} s_{n_k} = -\infty$.

4.2.4. If $\epsilon > 0$, there is an integer N such that $|s - s_n| < \epsilon$ if $n \geq N$. There is an integer K such that $n_k \geq N$ if $k \geq K$. Therefore, if $k \geq K$, then $|s - (-1)^k s_{n_k}| <$

4.2.5. Suppose that $|s| > 0$. Since $\lim_{n\to\infty} |s_n| = |s|$, there is an integer N such that $|s_n| > |s|/2$ if $n \geq N$ (Exercise 4.1.19). Since $|s - s_n| = |s| + |s_n|$ if s and s_n have the same sign, $|s - s_n| > 3|s|/2$ for infinitely many n. This contradicts the definition of $\lim_{n\to\infty} s_n = s$. Therefore, $s = 0$.

4.2.6. If $\{s_n\}$ is nonincreasing, then $\{s_{n_k}\}$ is also, so it suffices to show that (A) $\inf\{s_{n_k}\} = \inf\{s_n\}$ and apply Theorem 4.1.6**(b)**. Since the set of terms of $\{s_{n_k}\}$ is contained in the set of terms of $\{s_n\}$, (B) $\inf\{s_n\} \leq \inf\{s_{n_k}\}$. Since $\{s_n\}$ is noninceasing, there is for every n an integer $n_k > n$ such that $s_n \geq s_{n_k}$. This implies that $\inf\{s_n\} \geq \inf\{s_{n_k}\}$. This and (B) imply the conclusion.

4.2.7. **(b)** Choose $n_1, n_2, \dots,$ so that $n_1 < n_2 < \cdots < n_k < \cdots$ and $x_{n_k} > k$. (If this were impossible for some k, then $\{x_n\}$ would be bounded above.)

(c) Choose $n_1, n_2, \dots,$ so that $n_1 < n_2 < \cdots < n_k < \cdots$ and $x_{n_k} < -k$. (If this were impossible for some k, then $\{x_n\}$ would be bounded below.)

4.2.8. Since $\{s_n\}$ is bounded, $\{s_n\}$ has a convergent subsequence (Theorem 4.2.5)**(a)**. Let s be the limit of this convergent subsequence. If $\{s_n\}$ does not converge to s, there is an $\epsilon_0 > 0$ and a subsequence $\{s_{n_k}\}$ such that (A) $|s_{n_k} - s| \geq \epsilon_0$ $k \geq 1$. Since $\{s_{n_k}\}$ is bounded, $\{s_{n_k}\}$ has a convergent subsequence which must also converge to s, by assumption. This contradicts (A); hence, $\lim_{n\to\infty} s_n = s$.

For the counterexample, let $\{t_n\}$ be any convergent sequence (with $\lim_{n\to\infty} t_n = t$) and $\{s_n\} = \{t_1, 1, t_2, 2, \dots, t_n, n, \dots\}$, which does not converge. A convergent $\{s_{n_k}\}$ must be bounded; hence, there is an integer K such that $\{s_{n_k}\}_{k=K}^{\infty}$ is subsequence of $\{t_n\}$, and therefore $\lim_{k\to\infty} s_{n_k} = t$.

4.2.9. Let $\epsilon > 0$. If $\lim_{x\to\overline{x}} f(x) = L$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < |x - \overline{x}| < \delta$. If $\{x_n\}$ is any sequence of points in $N_{\overline{x}}$ such that $\lim_{n\to\infty} x_n = \overline{x}$, there is an integer N such that $0 < |x_n - \overline{x}| < \delta$ if $n \geq N$. Then $|f(x_n) - L| < \epsilon$ if $n \geq N$, so $\lim_{n\to\infty} f(x_n) = L$.

For sufficiency, suppose that $\lim_{x\to\overline{x}} f(x)$ does not exist, or exists and differs from L. Then there is an $\epsilon_0 > 0$ such that, for each integer n, there is an x_n in $N_{\overline{x}}$ that satisfies $|f(x_n) - L| \geq \epsilon_0$. Then $\lim_{n\to\infty} x_n = \overline{x}$ but $\lim_{n\to\infty} f(x_n)$ either does not exist or exists and differs from L.

4.2.10. Choose $\{x_n\}$ in $[a, b]$ so that $f(x_n) \leq \min(\alpha + 1/n, f(x_{n-1}))$, $n \geq 2$. Then

$\{f(x_n)\}_2^\infty$ is nonincreasing and $\inf\{f(x_n)\}_{n=2}^\infty = \alpha$; hence, (A) $\lim_{n\to\infty} f(x_n) = \alpha$ (Theorem 4.1.6)**(b)**. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ (Theorem 4.2.5(**a**)). If $\underline{x} = \lim_{k\to\infty} x_{n_k}$, then $\lim_{k\to\infty} f(x_{n_k}) = \alpha$, because of (A) and Theorem 4.2.2. Moreover, $\underline{x} \in [a,b]$ and f is continuous on $[a,b]$, so $\lim_{k\to\infty} f(x_{n_k}) = f(\underline{x})$ (Theorem 4.2.6). Hence, $f(\underline{x}) = \alpha$.

Now choose $\{x_n\}$ in $[a,b]$ so that $f(x_n) \ge \max(\beta - 1/n, f(x_{n-1}))$, $n \ge 2$. Then $\{f(x_n)\}_2^\infty$ is nondecreasing and $\sup\{f(x_n)\}_{n=2}^\infty = \beta$; hence, (B) $\lim_{n\to\infty} f(x_n) = \beta$ (Theorem 4.1.6)(**a**). Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ (Theorem 4.2.5(**a**)). If $\overline{x} = \lim_{k\to\infty} x_{n_k}$, then $\lim_{k\to\infty} f(x_{n_k}) = \beta$, because of (B) and Theorem 4.2.2. Moreover, $\overline{x} \in [a,b]$ and f is continuous on $[a,b]$, so $\lim_{k\to\infty} f(x_{n_k}) = f(\overline{x})$ (Theorem 4.2.6). Hence, $f(\overline{x}) = \beta$.

4.2.11. If f is not uniformly continuous on $[a,b]$, then, for some $\epsilon_0 > 0$, there are sequences $\{x_n\}$ and $\{y_n\}$ in $[a,b]$ such that $|x_n - y_n| < 1/n$ and (A) $|f(x_n) - f(y_n)| \ge \epsilon_0$. By Theorem 4.2.6(**a**), $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to a limit $\overline{x}$ in $[a,b]$. Since $|x_{n_k} - y_{n_k}| < 1/n_k$, $\lim_{k\to\infty} y_{n_k} = \overline{x}$ also. Then $\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(\overline{x})$ (Theorem 4.2.6), which contradicts (A).

4.2.12. Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \overline{x}$ and let $\lim_{n\to\infty} f(x_n) = L$. If $\{z_n\} = \{x_1, y_1, x_2, y_2, x_3, y_3, \dots\}$, then $\lim_{n\to\infty} z_n = \overline{x}$, so $\lim_{n\to\infty} f(z_n)$ exists, by assumption. Since $\{f(x_n)\}$ is a subsequence of $\{f(z_n)\}$, Theorem 4.2.2 implies that $\lim_{n\to\infty} f(z_n) = L$. Since $\{f(y_n)\}$ is a subsequence of $\{f(z_n)\}$, $\lim_{n\to\infty} f(y_n) = L$. Now apply Exercise 4.2.9.

4.2.13. Apply Exercise 4.1.12 to $g(x) = \dfrac{f(x) - f(\overline{x})}{x - \overline{x}}$.

4.3 INFINITE SERIES OF CONSTANTS

4.3.1. Apply Theorem 4.1.8 to the partial sums of the series.

4.3.2. Apply Theorem 4.1.8 to the partial sums of the series.

4.3.3. **(a)** If $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$, and $a_i = b_i$ for $i \ge N$, then $A_n = B_n + C$ if $n \ge N$, where C is a constant.

(b) If $A_n = a_1 + \cdots + a_n$, and $B_n = b_1 + \cdots + b_n$, then $B_{n_k} = A_k$. If $\sum_1^\infty b_n = B$, then $\lim_{k\to\infty} A_k = \lim_{k\to\infty} B_{n_k} = B$ (Theorem 4.2.2), so $\sum_1^\infty a_n = B$. If $\sum_1^\infty a_n = A$ (finite) and $\epsilon > 0$, there is an integer K such that $|B_{n_k} - A| = |A_k - A| < \epsilon$ if $k \ge K$. Since $B_n = B_{n_k}$ if $n_k \le n < n_{k+1}$, $|B_n - A| < \epsilon$ if $n \ge n_K$. Therefore, $\sum_1^\infty b_n = A$. A similar argument applies if $A = \pm\infty$.

4.3.4. **(a)** Apply Theorem 4.3.5.

(b) No; $\sum \frac{1}{n} = \infty$, but $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+r} < \frac{r+1}{n} \to 0$ as $n \to \infty$.

4.3.5. If $\epsilon > 0$, there is an integer K such that $\left|\sum_k^m a_n\right| < \epsilon/2$ if $m \ge k \ge K$ (Theorem 4.3.5). Therefore, if $k \ge K$, $\overline{\lim}_{m\to\infty} \left|\sum_{n=k}^m a_n\right| \le \epsilon/2$. Since $\sum_{n=k}^\infty a_n$ converges, this means that $\left|\sum_{n=k}^\infty a_n\right| \le \epsilon/2 < \epsilon$.

4.3.6. **(a)** $\sum_{n=k}^m \frac{1}{n^p} \le \sum_{n=k}^m \int_{n-1}^n \frac{dx}{x^p} < \int_{k-1}^\infty \frac{dx}{x^p} = \frac{1}{p-1}\frac{1}{(k-1)^{p-1}}$. Since this holds for all $m \ge k$, $\sum_{n=k}^\infty \frac{1}{n^p} < \frac{1}{p-1}\frac{1}{(k-1)^{p-1}}$.

(b) By writing

$$(-1)^k \sum_{n=k}^m \frac{(-1)^n}{n} = \left[\left(\frac{1}{k} - \frac{1}{k+1}\right) + \left(\frac{1}{k+2} - \frac{1}{k+3}\right) + \cdots\right] > 0$$

and

$$(-1)^k \sum_{n=k}^m \frac{(-1)^n}{n} = \left[\frac{1}{k} - \left(\frac{1}{k+1} - \frac{1}{k+2}\right) - \left(\frac{1}{k+3} - \frac{1}{k+4}\right)\cdots\right] < \frac{1}{k},$$

we see that $\left|\sum_{n=k}^m \frac{(-1)^n}{n}\right| < \frac{1}{k}$. Since this holds for all $m \ge k$, $\left|\sum_{n=k}^\infty \frac{(-1)^n}{n}\right| < \frac{1}{k}$.

4.3.7. Use Theorem 4.3.9; note that $\sum_{n=k}^\infty b_n = \sum_{n=k-1}^\infty b_{n+1}$.

4.3.8. **(a)** $a_n = \frac{\sqrt{n^2-1}}{\sqrt{n^5+1}} = \frac{n\sqrt{1-1/n^2}}{n^{5/2}\sqrt{1+1/n^5}} = \frac{1}{n^{3/2}}\frac{\sqrt{1-1/n^2}}{\sqrt{1+1/n^5}}$. With $b_n = \frac{1}{n^{3/2}}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$; since $\sum b_n < \infty$, $\sum a_n < \infty$.

(b) With $a_n = \frac{1}{n^2[1 + \frac{1}{2}\sin(n\pi/4)]}$ and $b_n = \frac{1}{n^2}$, $\overline{\lim}_{n\to\infty} \frac{a_n}{b_n} = 2$. Since $\sum b_n < \infty$, $\sum a_n < \infty$.

(c) $a_n = \frac{1 - e^{-n}\log n}{n}$; let $b_n = \frac{1}{n}$. Since $\overline{\lim}_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum \frac{1}{n} = \infty$, $\lim_{n\to\infty} a_n = \infty$, by Corollary 4.3.12

(d) If $a_n = \cos\frac{\pi}{n^2}$, then $\lim_{n\to\infty} a_n = 1$ and $\sum a_n = \infty$ by Corollary 4.3.6

(e) $a_n = \sin\dfrac{\pi}{n^2}$; let $b_n = \dfrac{1}{n^2}$. Since $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \pi$ and $\sum b_n < \infty$, $\sum a_n < \infty$, by Corollary 4.3.12.

(f) $a_n = \dfrac{1}{n}\tan\dfrac{\pi}{n}$; let $b_n = \dfrac{1}{n^2}$. Since $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \pi$ and $\sum b_n < \infty$, $\sum a_n < \infty$, by Corollary 4.3.12.

(g) $\dfrac{1}{n}\cot\dfrac{\pi}{n}$; let $b_n = \dfrac{1}{n}$. Since $\lim_{n\to\infty}\dfrac{a_n}{b_n} = 1$ and $\sum b_n = \infty$, $\sum a_n = \infty$, by Corollary 4.3.12.

(h) $a_n = \dfrac{\log n}{n^2}$; let $b_n = \dfrac{1}{n^{3/2}}$. Since $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{\log n}{\sqrt{n}} = 0$ and $\sum b_n < \infty$, $\sum a_n < \infty$, by Theorem 4.3.11**(a)**.

4.3.9. If $\int^{\infty} f(x)\,dx < \infty$, there is an M such that

$$\int_k^n f(x)\,dx = \sum_{m=k}^{n}\int_{x_n}^{x_{n+1}} f(x)\,dx < M, \quad n \ge k, \tag{A}$$

(Theorem 3.4.5). Therefore, $\sum_{n=k}^{\infty}\int_n^{n+1} f(x)\,dx < M$, $n \ge k$, so the series converges (Theorem 4.3.8). Conversely, if the series converges, then (A) holds for some M (Theorem 4.3.8). If $T > k$, choose an integer N such that $N > T$. Then (A) implies that $\int_k^T f(x)\,dx \le \int_k^N f(x)\,dx \le M$, so $\int_k^{\infty} f(x)\,dx < \infty$ (Theorem 3.4.5).

All three series diverge if $p \le 0$, by Corollary 4.3.6. For $p > 0$ we consider the associated improper integrals.

4.3.10. **(a)** $\int^{\infty}\dfrac{x}{x^2-1}\,dx = \dfrac{1}{2}\log(x^2-1)\Big|^{\infty} = \infty$; if $p \ne 1$, $\int^{\infty}\dfrac{x}{(x^2-1)^p}\,dx = \dfrac{1}{2}\dfrac{(x^2-1)^{-p+1}}{-p+1}\Big|^{\infty}\begin{cases} = \infty & \text{if } 0 < p < 1, \\ < \infty & \text{if } p > 1.\end{cases}$ Therefore, the series converge if and only if $p > 1$.

(b) $\int^{\infty}\dfrac{x}{x^2-1}\,dx = \dfrac{1}{3}\log(x^3+4)\Big|^{\infty} = \infty$; if $p \ne 1$, $\int^{\infty}\dfrac{x}{(x^3+4)^p}\,dx = \dfrac{1}{3}\dfrac{(x^3+4)^{-p+1}}{-p+1}\Big|^{\infty}\begin{cases} = \infty & \text{if } 0 < p < 1, \\ < \infty & \text{if } p > 1.\end{cases}$ Therefore, the series converge if and only if $p > 1$.

(c) since $\lim_{n\to\infty}\dfrac{\sinh n}{\cosh n} = 1$, the series diverges if $p = 1$, by Corollary 4.3.6; if $p \ne 1$, $\int^{\infty}\dfrac{\sinh x}{(\cosh x)^p}\,dx = \dfrac{(\cosh n)^{-p+1}}{-p+1}\Big|^{\infty}\begin{cases} = \infty & \text{if } 0 < p < 1, \\ < \infty & \text{if } p > 1.\end{cases}$ Therefore, the series converge if and only if $p > 1$.

4.3.11. Use the integral test.

4.3.12. If $f = \dfrac{g}{g'}$, then $f' = \dfrac{gg'' - (g')^2}{g^2} < 0$. Apply the integral test with $f = \dfrac{g'}{g}$.

4.3.13. $\sum_{n=N+1}^{m}\int_{n}^{n+1}\frac{dx}{x^p} < \sum_{n=N+1}^{m}\frac{1}{n^p} < \sum_{n=N+1}^{m}\int_{n-1}^{n}\frac{dx}{x^p}$; hence, $\int_{N+1}^{m+1}\frac{dx}{x^p} < \sum_{n=N+1}^{m}\frac{1}{n^p} < \int_{N}^{m}\frac{dx}{x^p}$. Letting $m \to \infty$ yields $\int_{N+1}^{\infty}\frac{dx}{x^p} < \sum_{n=N+1}^{\infty}\frac{1}{n^p} < \int_{N}^{\infty}\frac{dx}{x^p}$, which implies the result.

4.3.14. **(a)** (A) $f(k+1) < \int_k^{k+1} f(x)\,dx < f(k)$. Since

$$a_n = f(1) + \sum_{k=1}^{n-1}\left[f(k+1) - \int_k^{k+1} f(x)\,dx\right],$$

the first inequality in (A) implies that $a_{n+1} < a_n < f(1)$. Also,

$$a_n = \sum_{k=1}^{n-1}\left[f(k) - \int_k^{k+1} f(x)\,dx\right] + f(n) > 0,$$

by the second inequality in (A). Therefore, $\lim_{n\to\infty} a_n = \inf\{a_n\}$, by Theorem 4.1.6**(b)**. Since $\sum_{k=1}^{n-1}\left[f(k) - \int_k^{k+1} f(x)\,dx\right]$ is an increasing function of n, $\inf\{a_n\} > 0$

(b) Take $f(x) = 1/x$.

4.3.15. **(a)** $a_n = \frac{2+\sin n\theta}{n^2+\sin n\theta}$; if $b_n = \frac{1}{n^2}$, then $\overline{\lim_{n\to\infty}}\frac{a_n}{b_n} = \overline{\lim_{n\to\infty}}\frac{2+\sin n\theta}{1+(\sin n\theta)/n^2} < 3$. Since $\sum b_n < \infty$, $\sum a_n < \infty$, by Theorem 4.3.11**(b)**.

(b) $a_n = \frac{n+1}{n}r^n$; if $b_n = r^n$, then $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$; since $\sum b_n$ converges if $0 < p < 1$ and diverges if $p \ge 1$, the same is true of $\sum a_n$, by Corollary 4.3.12.

(c) Since $\lim_{n\to\infty} e^{-n\rho}\cosh n\rho = \frac{1}{2}$ the series diverges, by Corollary 4.3.6.

(d) $a_n = \frac{n+\log n}{n^2(\log n)^2}$; if $b_n = \frac{1}{n(\log n)^2}$, then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{\log n}{n}\right) = 1$; since $\sum b_n < \infty$ by the integral test, $\sum a_n < \infty$, by Corollary 4.3.12.

(e) $a_n = \frac{n+\log n}{n^2\log n}$; if $b_n = \frac{1}{n\log n}$, then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{\log n}{n}\right) = 1$; since $\sum b_n = \infty$ by the integral test, $\sum a_n = \infty$, by Corollary 4.3.12.

(f) $a_n = \frac{(1+1/n)^n}{2^n}$; if $b_n = \frac{1}{2^n}$, then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$; since $\sum b_n < \infty$, $\sum a_n < \infty$, by Corollary 4.3.12.

4.3.16. The series diverges if $q_0 = q_1 = \cdots = q_m = 0$ (Exercise 4.3.13). Suppose that

$q_i = 0$ if $0 \le i < j$ and $q_j \ne 0$. Let a_n be the general term of the series and

$$b_n = \frac{1}{L_0(n)^{p+1}} \quad \text{if} \quad j = 0,$$

$$b_n = \frac{1}{L_0(n)\dots L_{j-1}(n)\left[L_j(n)\right]^{p+1}} \qquad \text{if} \qquad 1 \le j \le m,$$

where $pq_j > 0$ and $0 < |p| < |q_j|$. Then $\sum b_n = \infty$ if $p < 0$ and $\sum b_n < \infty$ if $p > 0$ (Exercise 4.3.11). From Exercise 2.4.42,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} 0 & \text{if} \quad q_j > 0, \\ \infty & \text{if} \quad q_j < 0. \end{cases}$$

Now use Theorem 4.3.11.

4.3.17. **(a)** $a_n = \dfrac{2+\sin^2(n\pi/4)}{3^n}$; if $b_n = \dfrac{1}{3^n}$, then $\overline{\lim_{n\to\infty}} \dfrac{a_n}{b_n} = \overline{\lim_{n\to\infty}} \left(2+\sin^2 \dfrac{n\pi}{2}\right) = 3$. Since $\sum b_n < \infty$, $\sum a_n < \infty$, by Theorem 4.3.11**(a)**.

(b) $a_n = \dfrac{n(n+1)}{4^n}$; $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{n+2}{4n} = \dfrac{1}{4}$; $\sum a_n < \infty$ by the ratio test.

(c) $a_n = \dfrac{3-\sin(n\pi/2)}{n(n+1)}$; if $b_n = \dfrac{1}{n^2}$, then $\overline{\lim_{n\to\infty}} \dfrac{a_n}{b_n} = \overline{\lim_{n\to\infty}} \dfrac{n}{n+1}\left(3-\sin\dfrac{n\pi}{2}\right) = 4$. Since $\sum b_n < \infty$, $\sum a_n < \infty$, by Theorem 4.3.11**(a)**.

(d) $a_n = \dfrac{n+(-1)^n}{n(n+1)} \dfrac{1}{2^n+\cos(n\pi/2)}$; if $b_n = \dfrac{1}{2^n}$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$. Since $\sum b_n < \infty$, $\sum a_n < \infty$, by Theorem 4.3.11**(a)**.

4.3.18. **(a)** $a_n = \dfrac{n!}{r^n}$; $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{n+1}{r} = \infty$; $\sum a_n = \infty$, by the ratio test.

(b) $a_n = n^p r^n$ $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} r\left(1+\dfrac{1}{n}\right)^p = r$; $\sum a_n$ converges if $0 < r < 1$, diverges if $r = 1$, by the ratio test. If $r = 1$, the series is $\sum n^p$, which converges if and only if $p < -1$.

(c) $a_n = \dfrac{r^n}{n!}$; $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{r}{n+1}$; $\sum a_n < \infty$, by the ratio test.

(d) $a_n = \dfrac{r^{2n+1}}{(2n+1)!}$; $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{r^2}{(2n+2)(2n+3)} = 0$; $\sum a_n < \infty$, by the ratio test.

(e) $a_n = \dfrac{r^{2n}}{(2n)!}$; $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{r^2}{(2n+1)(2n+2)} = 0$; $\sum a_n < \infty$, by the ratio test.

4.3.19. **(a)** $a_n = \dfrac{(2n)!}{2^{2n}(n!)^2}$; $\dfrac{a_{n+1}}{a_n} - 1 = \dfrac{2n+1}{2n+2} - 1 = -\dfrac{1}{2n+2}$; $\lim_{n\to\infty} n\left(\dfrac{a_{n+1}}{a_n} - 1\right) = -\dfrac{1}{2}$; $\sum a_n = \infty$, by Raabe's test.

(b) $a_n = \dfrac{(3n)!}{3^{3n} n!(n+1)!(n+3)!}$; $\dfrac{a_{n+1}}{a_n} - 1 = \dfrac{(3n+2)(3n+1)}{9(n+2)(n+4)} - 1 = -\dfrac{5(9n+4)}{9(n+2)(n+4)}$; $\lim_{n\to\infty} n\left(\dfrac{a_{n+1}}{a_n} - 1\right) = -5$; $\sum a_n < \infty$, by Raabe's test.

(c) $a_n = \dfrac{2^n n!}{5\cdots 7\cdot(2n+3)}$; $\dfrac{a_{n+1}}{a_n} - 1 = \dfrac{2(n+1)}{2n+5} - 1 = -\dfrac{3}{2n+5}$; $\lim_{n\to\infty} n\left(\dfrac{a_{n+1}}{a_n} - 1\right) = -\dfrac{3}{2}$; $\sum a_n < \infty$, by Raabe's test.

(d) $a_n = \dfrac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\beta(\beta+1)\cdots(\beta+n-1)}$ $(\alpha, \beta > 0)$; $\dfrac{a_{n+1}}{a_n} - 1 = \dfrac{\alpha+n}{\beta+n} - 1 = \dfrac{\alpha-\beta}{\beta+n}$; $\lim_{n\to\infty} n\left(\dfrac{a_{n+1}}{a_n} - 1\right) = \alpha - \beta$; convergent if $\alpha < \beta - 1$, divergent if $\alpha > \beta - 1$. If $\alpha = \beta - 1$, then $a_n = \dfrac{\beta - 1}{\beta + n - 1}$. If $b_n = \dfrac{1}{n}$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 1$; since $\sum b_n = \infty$, $\sum a_n = \infty$, by Corollary 4.3.12.

4.3.20. **(a)** $a_n = \dfrac{n^n\,(2+(-1)^n)}{2^n}$; $a_n^{1/n} = \dfrac{n}{2}(2+(-1)^n)^{1/n}$; $\lim_{n\to\infty} a_n^{1/n} = \infty$; $\sum a_n = \infty$, by Cauchy's root test.

(b) $a_n = \left(\dfrac{1+\sin 3n\theta}{3}\right)^n$; $a_n^{1/n} = \dfrac{1+\sin 3n\theta}{3}$; $\overline{\lim_{n\to\infty}}\, a_n^{1/n} \le \dfrac{2}{3}$; $\sum a_n < \infty$, by Cauchy's root test.

(c) $a_n = (n+1)\left(\dfrac{1+\sin(n\pi/6)}{3}\right)^n$; $a_n^{1/n} = (n+1)^{1/n}\left(\dfrac{1+\sin(n\pi/6)}{3}\right)$; $\overline{\lim_{n\to\infty}}\, a_n^{1/n} = \dfrac{2}{3}$; $\sum a_n < \infty$, by Cauchy's root test.

(d) $a_n = \left(1 - \dfrac{1}{n}\right)^{n^2}$; $a_n^{1/n} = \left(1 - \dfrac{1}{n}\right)^n$; $\lim_{n\to\infty} a_n^{1/n} = \dfrac{1}{e}$; $\sum a_n < \infty$, by Cauchy's root test.

4.3.22. Recall that if a series S_1 is obtained by dropping finitely many terms from a series S, then S and S_1 converge or diverge together. Let the given series be $S = \sum_{n=m}^{k} a_n$ where $m \le k$, and let $\widehat{S} = \sum_{n=m}^{k} |a_n|$. Since S converges, so does $S_1 = \sum_{n=k}^{\infty} a_n$. Let $\widehat{S}_1 = \sum_{n=m}^{\infty} |a_n|$; then $\widehat{S}_1 = \pm\widehat{S}_1$, so $\widehat{S}_1$ converges. But $\widehat{S}_1$ and $\widehat{S}$ converge or diverge together, so $\widehat{S}$ converges.

4.3.23. If the assertion were false, there would be an integer N_0 and a constant J_0 such that $\sum_{n=N_0}^{N_0+k} a_n \le J_0$ for all $k > 0$. But this implies that $\sum a_j < \infty$ (Theorem 4.3.8), a contradiction.

4.3.24. Suppose that $\sum_{n=m}^{\infty} |a_n| < \infty$. Let $b_n = |a_n| - a_n$; then $0 \le b_n \le 2|a_n|$, so $\sum_{n=m}^{\infty} b_n$ converges absolutely, by the comparison test. Since $a_n = |a_n| - b_n$, $\sum_{n=m}^{\infty} a_n$ converges, by Theorem 4.3.3.

4.3.25. **(a)** $a_n = (-1)^n \dfrac{1}{n(\log n)^2}$; $\sum |a_n| < \infty$, by the integral test, since $\int^{\infty} \dfrac{1}{x(\log x)^2}\,dx < \infty$.

(b) $a_n = \dfrac{\sin n\theta}{2^n}$; $\sum |a_n| < \infty$, by comparison with $\sum \dfrac{1}{2^n}$.

(c) $a_n = (-1)^n \dfrac{1}{\sqrt{n}} \sin \dfrac{\pi}{n}$; since $\left|\sin \dfrac{\pi}{n}\right| \le \dfrac{\pi}{n}$, $\sum |a_n| < \infty$, by comparison with $\sum \dfrac{\pi}{n^{3/2}}$.

(d) $a_n = \dfrac{\cos n\theta}{\sqrt{n^3 - 1}}$; if $b_n = \dfrac{1}{n^{3/2}}$, then $\overline{\lim_{n\to\infty}} \dfrac{|a_n|}{b_n} \le 1$. Since $\sum \dfrac{1}{n^{3/2}} < \infty$, $\sum |a_n| < \infty$, by Theorem 4.3.11**(a)**.

4.3.26. **(a)** $a_n b_n = \dfrac{n \sin n\theta}{n^2 + (-1)^n}$, with $a_n = \dfrac{n}{n^2 + (-1)^n}$ and $b_n = \sin n\theta$. The partial sums of $\sum b_n$ are bounded (shown in text), and

$$\begin{aligned} a_{n+1} - a_n &= \frac{n+1}{(n+1)^2 + (-1)^{n+1}} - \frac{n}{n^2 + (-1)^n} \\ &= \frac{(-1)^n(2n+1) - n(n+1)}{((n+1)^2 + (-1)^{n+1})(n^2 + (-1)^n)}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} n^2 |a_{n+1} - a_n| = 1$. Since $\sum \dfrac{1}{n^2} < \infty$, $\sum |a_{n+1} - a_n| < \infty$, by Theorem 4.3.11**(a)**. Now apply Dirichlet's test.

(b) $a_n b_n = \dfrac{\cos n\theta}{n}$ with $a_n = \dfrac{1}{n}$ and $b_n = \cos n\theta$. To be specific, consider $\sum_{n=1}^{\infty} \dfrac{\cos n\theta}{n}$. Since $\cos r\theta = \dfrac{\sin\left(r - \frac{1}{2}\right)\theta - \sin\left(r + \frac{1}{2}\right)\theta}{2\sin(\theta/2)}$ $(\theta \ne 2k\pi)$,

$$\begin{aligned} B_n &= \frac{(\sin\frac{3}{2}\theta - \sin\frac{5}{2}\theta) + (\sin\frac{5}{2}\theta - \sin\frac{7}{2}\theta) + \cdots + \left(\sin\left(n - \frac{1}{2}\right)\theta - \sin(n + \frac{1}{2})\theta\right)}{2\sin(\theta/2)} \\ &= \frac{\sin\frac{3}{2}\theta - \sin(n + \frac{1}{2})\theta}{2\sin(\theta/2)}, \end{aligned}$$

which implies that $|B_n| \le \left|\dfrac{1}{\sin(\theta/2)}\right|$, $n \ge 2$. Since $|a_{n+1} - a_n| = \dfrac{1}{n(n+1)} < \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n^2} < \infty$, the conclusion follows from Dirichlet's theorem.

4.3.27. **(a)** $a_n = \dfrac{1}{\sqrt{n}}$ is decreasing with n and $|B_n| \le 2$ for all n. Abel's test implies convergence. Since $\sum \dfrac{1}{\sqrt{n}} = \infty$, the convergence is conditional.

(b) $a_n = \dfrac{1}{n}$ is decreasing with n and $\{|B_n|\}$ is bounded (shown in text). Abel's test implies convergence. Now consider the series $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left|\sin\frac{n\pi}{6}\right|$. Since $\displaystyle\sum_{6m+1}^{6(m+1)} \frac{1}{n}\left|\sin\frac{n\pi}{6}\right| > \frac{5}{6(m+1)}\left|\sin\frac{\pi}{6}\right|$ for every m and $\sum \dfrac{1}{m} = \infty$, $S = \infty$, by Theorem 4.3.8. Hence, $\sum \dfrac{1}{n}\sin\dfrac{n\pi}{6}$ converges conditionally.

Since **(c)** $\left|\dfrac{1}{n^2}\cos\dfrac{n\pi}{6}\right| \le \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n^2} < \infty$, the given series converges absolutely.

(d) If $a_n = \dfrac{1\cdot 3\cdot 5\cdots(2n+1)}{4\cdot 6\cdot 8\cdots(2n+4)}$, then $\dfrac{a_{n+1}}{a_n} - 1 = -\dfrac{3}{2n+6}$; $\displaystyle\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n} - 1\right) = -\frac{3}{2}$, so $\sum a_n < \infty$ by Raabe's test, and the given series converges absolutely, by the comparison test.

4.3.28. Let $g(n) = \dfrac{a_0 + \cdots + a_k n^k}{b_0 + \cdots + b_s n^s}$, with $a_k, b_s \ne 0$. Then

$$\lim_{n\to\infty} |g(n)r^n|^{1/n} = |r| \lim_{n\to\infty} |g(n)|^{1/n} = |r|;$$

hence, the series converges absolutely if $|r| < 1$ (Theorem 4.3.17). If $|r| > 1$, then $\displaystyle\lim_{n\to\infty} |g(n)r^n| = \infty$, so the series diverges (Corollary 4.3.6). If $|r| = 1$, Theorem 4.3.11 implies absolute convergence if and only if $s \ge k+2$, since $\displaystyle\lim_{n\to\infty} |g(n)|n^{s-k} = \left|\frac{a_k}{b_s}\right|$. The series does not converge conditionally if $r = 1$, since its terms have the same sign for large n (Exercise 4.3.22); if $r = -1$, the series converges conditionally if and only if $s = k+1$ (Corollary 4.3.22).

4.3.29. Since $(a_n \pm b_n)^2 \ge 0$, $\pm 2a_n b_2 \le a_n^2 + b_n^2$, so $|a_n b_n| \le (a_n^2 + b_n^2)/2$. Apply the comparison test.

4.3.30. **(a)** If $\sum |a_n| < \infty$, then $|a_n| < 1$, and so $a_n^2 < |a_n|$ for large n, and the comparison test implies that $\sum a_n^2 < \infty$.

4.3.31. Let $B_n = \displaystyle\sum_{j=1}^{n} b_j$, $s_n = \displaystyle\sum_{j=1}^{n} a_j b_j$, and $t_n = \displaystyle\sum_{j=1}^{n} (a_j - a_{j+1})B_j$; then (A) $s_n = t_{n-1} + a_n B_n \ge t_{n-1}$ for n sufficiently large. From Theorem 4.3.11**(b)**, $\displaystyle\lim_{n\to\infty} t_n = \infty$, since $\displaystyle\lim_{j\to\infty} \frac{(a_j - a_{j+1})B_j}{(a_j - a_{j+1})w_j} = \lim_{j\to\infty} \frac{B_j}{w_j} > 0$. Now (A) implies that $\displaystyle\lim_{n\to\infty} s_n = \infty$.

4.3.32. **(a)** If $|\sin m\theta| \le \sin\epsilon$, then $|m\theta - j\pi| \le \epsilon$ for some integer j, and $j\pi + \epsilon <$

$(m+1)\theta < (j+1)\pi - \epsilon$ if $0 < 2\epsilon < \theta < \pi - 2\epsilon$; hence, $|\sin(m+1)\theta| > \sin\epsilon$.

(b) The series converges (Example 4.3.22). To see that it does not converge absolutely, assume without loss of generality that $0 < 2\epsilon < \theta < \pi - 2\epsilon$ and use Exercise 4.3.31, with $a_n = n^{-p}$, $b_n = |\sin n\theta|$, and $w_n = n$. From **(a)**, $\varliminf_{n\to\infty} \dfrac{B_n}{n} > 0$. Also, $a_n - a_{n+1}e \geq p(n+1)^{-p-1}$, so $\sum n(a_n - a_{n+1}) = \infty$.

4.3.33. Insert parentheses: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{m=1}^{\infty}\left(\frac{1}{2m-1} - \frac{1}{2m}\right)$.

4.3.34. Insert parentheses: $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{n} = \sum_{m=0}^{\infty}\left(\frac{1}{3m+1} - \frac{2}{3m+1} + \frac{1}{3m+3}\right)$.

4.3.35. Their partial sums are the same for sufficiently large n.

4.3.36. In all parts we use the notation introduced in the proof for finite μ and ν, and $m_0 = n_0 = 0$.

(a) Suppose that $k \geq 1$. Let m_k be an integer such that

$$m_k > m_{k-1} \quad \text{and} \quad \sum_{i=1}^{m_k} \alpha_i - \sum_{j=1}^{n_{k-1}} \beta_j \geq \mu + k. \tag{A}$$

Let n_k be the smallest integer such that

$$n_k > n_{k-1} \quad \text{and} \quad \sum_{i=1}^{m_k} \alpha_i - \sum_{j=1}^{n_k} \beta_j \leq \mu. \tag{B}$$

Then (A) implies that $\varlimsup_{n\to\infty} B_n = \infty$. Since n_k is the smallest integer for which (B) holds,

$$\mu - \beta_{n_k} < B_{m_k+n_k} \leq \mu, \quad k \geq 2. \tag{C}$$

Since $b_n < 0$ if $m_k + n_{k-1} < n \leq m_k + n_k$,

$$B_{m_k+n_k} \leq B_n \leq B_{m_k+n_{k-1}}, \quad m_k + n_{k-1} \leq n \leq m_k + n_k. \tag{D}$$

Since $b_n > 0$ if $m_k + n_k < n \leq m_{k+1} + n_k$,

$$B_{m_k+n_k} \leq B_n \leq B_{m_{k+1}+n_k}, m_k + n_k \leq n \leq m_{k+1} + n_k. \tag{E}$$

From the first inequalities in (C), (D), and (E),

$$B_n \geq \mu - \beta_{n_k}, \quad m_k + n_{k-1} \leq n \leq m_{k+1} + n_k. \tag{F}$$

From the second inequality in (C), $B_n \leq \mu$ for infinitely many values of n. However, since $\lim_{j\to\infty} \beta_j = 0$, (F) implies that if $\epsilon > 0$, then $B_n < \mu - \epsilon$ for only finitely many values of n. Therefore, $\varliminf_{n\to\infty} B_n = \mu$.

(b) Suppose that $k \geq 1$. Let m_k be an integer such that $m_k > m_{k-1}$ and $\displaystyle\sum_{i=1}^{m_k} \alpha_i - \sum_{j=1}^{n_{k-1}} \beta_j > k$. Then $\overline{\lim_{n\to\infty}} B_n = \infty$. Let n_k be an integer such that $n_k > n_{k-1}$ and $\displaystyle\sum_{i=1}^{m_k} \alpha_i - \sum_{j=1}^{n_k} \beta_j < -k$. Then $\underline{\lim_{n\to\infty}} B_n = -\infty$.

(c) Suppose that $k \geq 1$. Let m_k be an integer such that $m_k > m_{k-1}$ and $\displaystyle\sum_{i=1}^{m_k} \alpha_i - \sum_{j=1}^{n_{k-1}} \beta_j > k$. Let $n_k = k$. Then $B_{m_k+n_{k-1}} > k$ and $B_{m_k+n_k} > k - \beta_k$. Since $\lim_{k\to\infty} \beta_k = 0$, $\lim_{n\to\infty} B_n = \infty$,

4.3.37. It must have infinitely many nonnegative terms $\{a_i\}$ and infinitely many negative terms $\{-\beta_j\}$ such that $\sum \alpha_i = \sum \beta_j = \infty$ and $\lim_{i\to\infty} \alpha_i = \lim_{j\to\infty} \beta_j = 0$.

4.3.38. The series of positive terms must diverge to ∞ and the series of nonpositive terms must converge.

4.3.39. Let $a_n = \dfrac{f^{(n)}(0)}{n!}$ and $b_n = \dfrac{g^{(n)}(0)}{n!}$; then

$$c_n = \sum_{r=0}^{n} a_r b_{n-r} = \sum_{r=0}^{n} \frac{f^{(r)}(0)}{r!}\frac{g^{(n-r)}(0)}{(n-r)!} = \frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r} f^{(r)}(0)g^{(n-r)}(0) = \frac{h^{(n)}(0)}{n!},$$

from Exercise 2.3.12.

4.3.40. Let $K = \sum_{r=0}^{\infty} |a_r|$.

$$\begin{aligned}|C_n - A_nB_n| &= |C - A_nB + A_n(B - B_n)| \leq |C - A_nB_n| + |A_n||B - B_n| \\ &\leq |C - A_nB| + K|B - B_n|.\end{aligned}$$

Given $\epsilon > 0$, choose N_1 so that $|B - B_n| < \epsilon$ if $n \leq N_1$. Then

$$|C_n - A_nB_n| < |C - A_nB| + K\epsilon, \quad n \geq N_1. \tag{A}$$

$$c_n = \sum_{m=0}^{r}\left(\sum_{r=0}^{m} a_r b_{m-r}\right) = \sum_{r=0}^{n} a_r \sum_{m=r}^{n} b_{m-r} = \sum_{r=0}^{n} a_r B_{n-r};$$

therefore,

$$C_n - A_nB = \sum_{r=0}^{n} a_r(B_{n-r} - B). \tag{B}$$

Choose $N_2 \geq N_1$ so that $\displaystyle\sum_{r=N_2}^{n} |a_r| < \epsilon$ if $n \geq N_2$ (Theorem 4.3.5). Since $\{B_n\}$ converges, $\{B_n\}$ is bounded. Let M be a constant such that $|B_n| + |B| \leq M$ for $n \geq 0$. From (B),

$$|C_n - A_nB| \leq \sum_{r=0}^{N_2-1} |a_r||B_{n-r} - B| + M\epsilon, \quad n \geq N_2. \tag{C}$$

Choose $N_3 \ge N_2$ so that $|B_j - B| < \epsilon$ if $j > N_3 - N_2 + 1$. If $n \ge N_3$ and $0 \le r \le N_2 - 1$, then $n - r > N_3 - N_2 + 1$, (C) implies that

$$|C_n - A_n B| < \epsilon \left(M + \sum_{r=0}^{N_2-1} |a_r| \right) \le (M + K\epsilon).$$

This and (A) imply $|C_n - A_n B_n| < (M + 2K)\epsilon$, $n \ge N_3$. Therefore, $\lim_{n\to\infty} C_n = \lim_{n\to\infty} A_n B_n = AB$

4.3.41. Denote $\alpha_n = \frac{1}{n} \sum_{r,s=0}^{n-1} a_{r+s}$; then

$$\alpha_n = \frac{1}{n}\sum_{r=0}^{n-1}(r+1)a_r + \frac{1}{n}\sum_{r=n}^{2n-2}(2n-r-1)a_r < \frac{1}{n}\sum_{r=0}^{n-1}(r+1)a_r + \sum_{r=n}^{\infty} a_r.$$

If $\epsilon > 0$, choose N so that $\sum_{r=n}^{\infty} a_r < \epsilon$ if $n \ge N$ (Corollary 4.3.7). Then

$$\alpha_n < \frac{1}{n}\sum_{r=0}^{N-1}(r+1)a_r + 2\epsilon, n > N.$$

Now choose $N_1 \ge N$ so that

$$\frac{1}{n}\sum_{r=0}^{N-1}(r+1)a_r < \epsilon, \quad n > N_1.$$

Then $\alpha_n < 3\epsilon$ if $n > N_1$, so $\lim_{n\to\infty} \alpha_n = 0$.

Denote $\beta_n = \frac{1}{n} \sum_{r,s=0}^{n-1} a_{r-s}$.

$$\beta_n = a_0 + \frac{2}{n}\sum_{r=1}^{n-1}(n-r)a_r = a_0 + 2\sum_{r=1}^{n-1} a_r - \frac{2}{n}\sum_{r=1}^{n-1} ra_r = 2A - a_0 - 2\gamma_n,$$

where $\gamma_n = \sum_{r=n}^{\infty} a_r + \frac{1}{n}\sum_{r=1}^{n-1} ra_r$. If $\epsilon > 0$, choose N so that $\sum_{r=n}^{\infty} a_r < \epsilon$ if $n \ge N$. Then $\frac{1}{n}\sum_{r=1}^{n-1} ra_r < \frac{1}{n}\sum_{r=1}^{N-1} ra_r + \epsilon, n > N$. Now choose $N_1 \ge N$ so that $\frac{1}{n}\sum_{r=1}^{N} ra_r < \epsilon, n > N_1$. Then $\gamma_n < 3\epsilon$ if $n \ge N_1$, so $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} \beta_n = 2A - \alpha_0$.

4.3.42. Since (A) $|a_j^{(i)}| \le \sigma_j$, $i, j \ge 1$, and $a_j = \lim_{i\to\infty} a_j^{(i)}$, it follows that $|a_j| \le \sigma_j$, $j \ge 1$. Since $\sum \sigma_j < \infty$, (A), (B) and the comparison test imply that $\sum_{j=1}^{\infty} |a_j^{(i)}| < \infty$ and $\sum_{j=1}^{\infty} |a_j| < \infty$. If $N \ge 1$, then

$$\begin{aligned}\left|\sum_{j=1}^{\infty} a_j^{(i)} - \sum_{j=1}^{\infty} a_j\right| &\le \sum_{j=1}^{N} |a_j^{(i)} - a_j| + \sum_{j=N+1}^{\infty} |a_j^{(i)}| + \sum_{j=N+1}^{\infty} |a_j| \\ &\le \sum_{j=1}^{N} |a_j^{(i)} - a_j| + 2\sum_{j=N+1}^{\infty} \sigma_j .\end{aligned} \qquad \text{(A)}$$

Given $\epsilon > 0$, choose N so that $\sum_{j=N+1}^{\infty} \sigma_j < \epsilon$. Having fixed N, choose I so that $|a_j^{(i)} - a_j| < \frac{\epsilon}{N}$, for $1 \le j \le N$ if $i \ge I$. Then (A) implies that $\left|\sum_{j=1}^{\infty} a_j^{(i)} - \sum_{j=1}^{\infty} a_j\right| < 3\epsilon$ if $i \ge I$, which completes the proof.

4.4 SEQUENCES AND SERIES OF FUNCTIONS

4.4.1. **(a)** If $|x| > 1$, then $\{F_n(x)\}$ diverges. If $|x| < 1$ then $\lim_{n\to\infty} F_n(x) = (1 - x^2)\lim_{n\to\infty} x^n = 0$. Since $F_n(1) = F_n(-1) = 0$ for all n, $\lim_{n\to\infty} F_n(x) = 0$ if $|x| \le 1$.

(b) If $|x| > 1$, then $\{F_n(x)\}$ diverges. If $|x| < 1$ then $\lim_{n\to\infty} F_n(x) = (1 - x^2)\lim_{n\to\infty} nx^n = 0$. Since $F_n(1) = F_n(-1) = 0$ for all n, $\lim_{n\to\infty} F_n(x) = 0$ if $|x| \le 1$.

(c) If $|x| > 1$, then $\{F_n(x)\}$ diverges. If $|x| < 1$ then

$$\lim_{n\to\infty} F_n(x) = \left(\lim_{n\to\infty} x^n\right)\left(\lim_{n\to\infty}(1 - x^n)\right) = 0 \cdot 1 = 0.$$

Since $F_n(1) = 0$ for all n, $\lim_{n\to\infty} F_n(1) = 0$. Since $F_n(-1) = 0$ if n is even and $F_n(-1) = -2$ if n is odd, $\{F_n(-1)\}$ diverges. Therefore, $\lim_{n\to\infty} F_n(x) = 1$, $-1 < x \le 1$.

(d) By the mean value theorem, $F_n(x) = \sin x + \frac{x}{n}\cos\theta(x,n)$ where $\theta(x,n)$ is between x and $x + \frac{x}{n}$. Therefore, $|F_n(x) - \sin x| \le \frac{|x|}{n} \to 0$ as $n \to \infty$, so $\lim_{n\to\infty} F_n(x) = \sin x$, $-\infty < x < \infty$.

(e) $F_n(-1) = 0$ if n is odd and $F_n(-1) = 1$ if n is even; hence, $\{F_n(-1)\}$ diverges. Since $F_n(1) = 1$ for all n, $\lim_{n\to\infty} F_n(1) = 1$. Since $\lim_{n\to\infty} x^n = \lim_{n\to\infty} x^{2n} = 0$ if $|x| < 1$,

$\lim_{n\to\infty} F_n(x) = 0$ if $|x| < 1$. If $|x| > 1$, then

$$\lim_{n\to\infty} F_n(x) = \left(\lim_{n\to\infty} \frac{1+x^n}{x^{2n}}\right)\left(\lim_{n\to\infty} \frac{1}{1+1/x^{2n}}\right) = 0\cdot 1 = 0.$$

Therefore, $\lim_{n\to\infty} F_n(x) = \begin{cases} 1, & -1 < x \le 1, \\ 0, & |x| > 1. \end{cases}$

(f) From Taylor's theorem, $\sin\frac{x}{n} = \frac{x}{n} - \cos\theta(x,n)\frac{x^3}{6n^3}$ where $\theta(x,n)$ is between zero and $\frac{x}{n}$. Therefore, $|F_n(x) - x| \le \frac{x^2}{6n^2}$, so $\lim_{n\to\infty} F_n(x) = x$, $-\infty < x < \infty$.

(g) From Taylor's theorem, $1-\cos\frac{x}{n} = \frac{x^2}{n^2} - \cos\theta(x,n)\frac{x^4}{24n^4}$ where $\theta(x,n)$ is between zero and $\frac{x}{n}$. Therefore, $|F_n(x) - x^2| \le \frac{x^4}{24n^2}$, so $\lim_{n\to\infty} F_n(x) = x^2$, $-\infty < x < \infty$.

(h) Since $F_n(0) = 0$ for all n, $\lim_{n\to\infty} F_n(0) = 0$. If $x \ne 0$, then $\lim_{n\to\infty} F_n(x) = x \lim_{n\to\infty} \frac{n}{e^{nx^2}} = x\cdot 0 = 0$. Hence, $F(x) = 0$, $-\infty < x < \infty$.

(i) $F_n(x) = \frac{x^2+2nx+n^2}{x^2+n^2} = 1 + \frac{2nx}{x^2+n^2}$, so $|F_n(x) - 1| = \frac{2n|x|}{x^2+n^2} \le \frac{2|x|}{n}$; therefore, $\lim_{n\to\infty} F_n(x) = 1$, $-\infty < x < \infty$.

4.4.2. If $x_2 > x_1$, then $F_n(x_2) - F_n(x_1) \ge 0$ for all n; therefore, $F(x_2) - F(x_1) = \lim_{n\to\infty}(F_n(x_2) - F_n(x_1)) \ge 0$ (Exercise 4.1.1).

4.4.3. $F_n(x) = 1$ for only finitely many (say k) values of x in $[a,b]$, and is zero otherwise; hence, if σ is a Riemann sum of f over a partition P of $[a,b]$, then $|\sigma| \le k\|P\|$; hence, $\int_a^b F_n(x)\,dx = 0$. F is not integrable on $[a,b]$, from Example 3.1.5 and Theorem 3.2.2

4.4.4. If $x \in S$, $|(g+h)(x)| \le |g(x)| + |h(x)| \le \|g\|_S + \|h\|_S$; hence, (A) $\|g+h\|_S \le \|g\|_S + \|h\|_s$. Also, $|(gh)(x)| = |g(x)|\,|h(x)| \le \|g\|_S\|h\|_S$, so $\|gh\|_S \le \|g\|_S\|h\|_S$. Now suppose that either g or h is bounded on S. Replacing g by $g-h$ in (A) yields $\|g\| \le \|g-h\| + \|h\|$, so (B) $\|g-h\| \ge \|g\| - \|h\|$. Interchanging g and h here yields $\|h-g\| \ge \|h\| - \|g\|$, which is equivalent to (C) $\|g-h\| \ge \|h\| - \|g\|$, since $\|h-g\| = \|g-h\|$. Since

$$\big|\|g\| - \|h\|\big| = \begin{cases} \|g\| - \|h\| & \text{if} \quad \|g\| > \|h\|, \\ \|h\| - \|g\| & \text{if} \quad \|h\| > \|g\|, \end{cases}$$

(B) and (C) imply that $\|g-h\| \ge \big|\|g\| - \|h\|\big|$.

4.4.5. **(a)** $|F(x)| \le |x|^n$; since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$, $F(x) = 0$. If S_c is a closed subset of $(-1,1)$, then $S_c \subset [-r,r]$ for some r with $0 < r < 1$, so $|F_n(x) - F(x)| \le r^n$ if $x \in S_c$. Therefore, $\|F_n - F\|_{S_c} \le r^n$, so $\lim_{n\to\infty} \|F_n - F\| = 0$ and the convergence is uniform on S_c. To show that the convergence is not uniform on S, choose ρ so that (A) $0 < \rho < e^{-\pi}$. By L'Hospital's rule, $\lim_{x\to 0} \frac{1-\rho^x}{x} = -\log\rho$, so $\lim_{n\to\infty} n(1-\rho^{1/n}) = -\log\rho > \pi$

from (A). Therfore the interval $(\rho^{1/n}, 1)$ has length $> \dfrac{\pi}{2n}$ for n sufficiently large. Hence, $|\sin nx_n| = 1$ for some x_n in $(\rho^{1/n}, 1)$ and $|F_n(x_n)| \ge \rho$. Hence, $\|F_n - F\|_S \ge \rho$ for large n, and the convergence is not uniform on S.

(b) $F(x) = 1$ if $|x| < 1$; $F(x) = 0$ if $|x| > 1$. Hence, (A) $|F_n(x) - F(x)| = \dfrac{x^{2n}}{1 + x^{2n}}$ if $|x| < 1$ and (B) $|F_n(x) - F(x)| = \dfrac{1}{1 + x^{2n}}$ if $|x| > 1$. From (A), $|F_n(x) - F(x)| \le r^{2n}$ if $|x| \le r < 1$; from (B), $|F_n(x) - F(x)| \le R^{-2n}$ if $|x| \ge R > 1$. This implies uniform convergence on closed subsets of S. Letting $x \to 1-$ in (A) or $x \to 1+$ in (B) yields $\|F_n - F\|_S = \dfrac{1}{2}$ for every n, so the convergence is not uniform on S.

(c) $F_n(x) = \dfrac{\sin x}{x + 1/n^2}$, so $F(x) = \dfrac{\sin x}{x}$. Therefore, (A) $|F(x) - F_n(x)| = \left|\dfrac{\sin x}{x(1 + n^2 x)}\right|$. Since $\left|\dfrac{\sin x}{x}\right| < 1$ (Exercise 2.3.19), (A) implies that $|F(x) - F_n(x)| < \dfrac{1}{n^2 r}$ if $|x| \ge r > 0$, so the convergence is uniform on $[r, \infty)$ for every $r > 0$; however, letting $x \to 0+$ in (A) shows that $\|F_n - F\|_S = 1$ for every n, so the convergence is not uniform on S.

4.4.6. **(a)** If $S_1 \subset S$, then $\|F - F_n\|_{S_1} \le \|F - F_n\|_S$. Since $\lim_{n\to\infty} \|F - F_n\|_S = 0$, $\lim_{n\to\infty} \|F - F_n\|_{S_1} = 0$.

(b) If $\|F - F_n\|_{S_i} \le \epsilon$ for $n \ge N_i$ and $S = \cup_{k=1}^m S_k$, then $\|F - F_n\|_S \le \max\left\{\|F - F_n\|_{S_i} \,\middle|\, 1 \le i \le m\right\} < \epsilon$ if $n \ge N = \max(N_1, \dots, N_M)$.

(c) Let $F_n(x) = x^n$ and $S_k = \left(-\dfrac{k}{k+1}, \dfrac{k}{k+1}\right)$, $k = 1, 2, \dots$. Then $\|F_n\|_{S_k} = \left(\dfrac{k}{k+1}\right)^n \to 0$ as $n \to \infty$, so $\{F_n\}$ converges uniformly to zero on each S_k. However $\bigcup_{k=1}^{\infty} S_k = (-1, 1)$ and $\|F_n\|_{(-1,1)} = 1$ for every n.

4.4.7. **(a)** From Exercise 4.4.1**(a)**, $F(x) = 0$ on $S = [-1, 1]$. Since $F_n'(x) - F'(x) = x^{n-1}(n - (n+2)x^2)$, $F_n(x) - F(x)$ assumes its maximum value in $[-1, 1]$ at $x = \pm\left(\dfrac{n}{n+2}\right)^{1/2}$. Therefore, $\|F_n - F\|_{[-1,1]} = \dfrac{2}{n+2}\left(\dfrac{n}{n+2}\right)^{n/2} \to 0$ as $n \to \infty$, so the convergence is uniform on $[-1, 1]$.

(b) From Exercise 4.4.1**(b)**, $F(x) = 0$ on $S = [-1, 1]$. Since $F_n'(x) - F'(x) = nx^{n-1}(n - (n+2)x^2)$, $F_n(x) - F(x)$ assumes its maximum value in $[-1, 1]$ at $x = \pm\left(\dfrac{n}{n+2}\right)^{1/2}$. Therefore, $|F_n(x) - F(x)| \le nr^n$ if $|x| \le r$, but

$$\|F_n - F\|_S = 2\left(\frac{n}{n+2}\right)^{(n+2)/2} \to \frac{2}{e}$$

as $n \to \infty$. Hence, the convergence is uniform on $[-r, r] \cup \{1\} \cup \{-1\}$ if $0 < r < 1$, but not on $S = [-1, 1]$.

(c) From Exercise 4.4.1**(c)**, $F(x) = 0$ on $S = (-1, 1]$. $|F_n(x) - F(x)| \le 2r^n$ if $|x| \le r$, so the convergence is uniform on $[-r, r]$ if $0 < r < 1$. Since $F_n(2^{-1/n}) - F(2^{-1/n}) = \dfrac{1}{4}$ and $\|F_{2m+1} - F\|_{[-1,r]} \ge 2$, it is not uniform on $[r, 1]$ or $[-1, -r]$ for any r. Since $F_n(1) - F(0) = 0$ for all n, the convergence is uniform on $[-r, r] \cup \{1\}$ for all r in $(0, 1)$.e

(d) From Exercise 4.4.1**(d)**, $F(x) = \sin x$ and $|F_n(x) - F(x)| \le \dfrac{|x|}{n}$, $-\infty < x < \infty$; therefore, $\|F_n - F\|_{[-r,r]} \le \dfrac{|r|}{n}$, so the convergence is uniform on any bounded set. It is not uniform on $(-\infty, \infty)$ since, for example, $\left|F_n\left(\pm\dfrac{n\pi}{2}\right) - F\left(\pm\dfrac{n\pi}{2}\right)\right| = 1$.

(e) From Exercise 4.4.1**(a)**, $F(x) = \begin{cases} 1 & , -1 < x \le 1, \\ 0 & , |x| > 1. \end{cases}$

$$|F_n(x) - F(x)| = \frac{x^n(1 - x^n)}{1 + x^{2n}} \le 2r^n$$

if $|x| \le r < 1$ and $|F_n(x) - F(x)| = \left|\dfrac{1 + x^n}{1 + x^{2n}}\right| \le 2r^n$ if $|x| \ge \dfrac{1}{r}$, the convergence is uniform on $[-r, r] \cup (-\infty, -1/r] \cup [1/r, \infty)$, $0 < r < 1$. Since $\lim_{x\to -1+} |F_{2m+1}(x) - F(x)| = 1$ and $\lim_{x\to 1+} |F_n(x) - F(x)| = 1$, the convergence is not uniform on any set for which 1 or -1 is a limit point.

(f) From Exercise 4.4.1**(f)**, $F(x) = x$ and $|F_n(x) - F(x)| \le \dfrac{|x|^3}{6n^2}$, $-\infty < x < \infty$; therefore, $\|F_n - F\|_{[-r,r]} \le \dfrac{|r|^3}{n}$, so the convergence is uniform on any bounded set. Since $|F_n(n\pi) - F(n\pi)| = n\pi$, it is not uniform on $(-\infty, \infty)$.

(g) From Exercise 4.4.1**(g)**, $F(x) = x^2$ and $|F_n(x) - F(x)| \le \dfrac{x^4}{24n^2}$, $-\infty < x < \infty$; therefore, $\|F_n - F\|_{[-r,r]} \le \dfrac{|r|^4}{24n^2}$, so the convergence is uniform on any bounded set. Since $|F_n(2n\pi) - F(2n\pi)| = 2n^2\pi^2$, the convergence is not uniform on $(-\infty, \infty)$.

(h) From Exercise 4.4.1**(h)**, $F(x) = 0, -\infty < x < \infty$. Since $F_n'(x) - F'(x) = ne^{-nx^2}(1 - 2nx^2)$, $|F_n(x) - F(x)| = n|x|e^{-nx^2}$ is a decreasing function of $|x|$ if $|x| > (2n)^{-1/2}$. Therefore, $|F_n(x) - F(x)| \le nre^{-nr^2}$ if $|x| \ge r$ and $n > 1/2r^2$, so the convergence is uniform on $(-\infty, r] \cup [r, \infty)$ if $r > 0$. Since $|F_n(x) - F(x)| = \sqrt{n/2}\,e^{-1/2}$ when $|x| = (2n)^{-1/2}$, the convergence is not uniform on $(-\infty, \infty)$.

(i) From Exercise 4.4.1**(i)**, $F(x) = 1, -\infty < x < \infty$. $|F_n(x) - 1| = \dfrac{|2xn|}{x^2 + n^2} \le \dfrac{2r}{n}$ if $|x| \le r$, so the convergence is uniform on $[-r, r]$. Since $F_n(n) - 1 = 1$, it is not uniform on $(-\infty, \infty)$.

4.4.8. The Heine–Borel theorem implies that $[a, b] \subset \cup_{i=1}^{m} I_{x_i}$ for some $x_1, x_2, \dots, x_m$. Use Exercise 4.4.6**(b)**.

4.4.9. Suppose first that F_n is bounded on S if $n \ge N$. Then $|\|F_n\|_S - \|F\|_S| \le \|F_n -$

$F\|_S$, $n \ge N$ (Lemma 4.4.2), and $\lim_{n\to\infty} \|F_n\| = \|F\|$ because $\lim_{n\to\infty} \|F_n - F\|_S$=0 Now suppose that there are infinitely many integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\|F_{n_k}\|_S = \infty$. Since $\{F_n\}$ converges uniformly on S, Theorem 4.4.6 implies that $\|F_n\|_S = \infty$ for all sufficently large n. Therefore, $\|F\|_S = \infty$.

4.4.10. Since $\{F_n\}$ converges uniformly to F on S, there is an N_1 such that $\|F_n - F\|_S < 1$ if $n \ge N_1$. Suppose that F is bounded on S and $n \ge N_1$. Then $\|F_n\|_S = \|F + (F_n - F)\|_S \le \|F\|_S + \|F_n - F\|_S < \|F\|_S + 1$, so $\overline{\lim}_{n\to\infty} \|F_n\|_S \le \|F\|_S + 1$. Suppose that $\overline{\lim}_{n\to\infty} \|F_n\|_S < \infty$. Then there is an integer N_2 and a number M such that $\|F_n\|_S < M$ if $n \ge N_2$. Now choose $n \ge \max(N_1, N_2)$. Then $\|F\|_S = \|F_n + (F - F_n)\| \le \|F_n\| + \|F - F_n\| < M + 1$; that is, F is bounded on S.

4.4.11. Given $\epsilon > 0$ there are integers N_1 and N_2 such that $\|F_n - F\|_S < \epsilon$ if $n \ge N_1$ and $\|G_n - G\|_S < \epsilon$ if $n \ge N_2$. Therefore, $\|(F_n + G_n) - (F + G)\|_S \le \|F_n - F\|_S + \|G_n - G\|_S < 2\epsilon$ if $n \ge \max(N_1, N_2)$.

4.4.12. **(a)** Given $\epsilon > 0$ there are integers N_1 and N_2 such that $\|F_n - F\|_S < \epsilon$ if $n \ge N_1$ and $\|G_n - G\|_S < \epsilon$ if $n \ge N_2$. Therefore, $\|F_nG_n - FG\|_S = \|(F_nG_n - F_nG) + (F_nG - FG)\|_S \le \|F_n\|_S\|G_n - G\|_S + \|G\|_S\|F_n - F\|_S$. From Exercise 4.4.9, there is an integer N_3 and a constant M such that $\|F_n\|_S < M$ if $n \ge N_3$. If $n \ge \max(N_1, N_2, N_3)$, then $\|F_nG_n - FG\|_S < (M + \|G\|_S)\epsilon$.

4.4.13. **(a)** $|L_n - L_m| \le |L_n - F_n(x)| + |F_n(x) - F_m(x)| + |F_m(x) - L_m|$. If $\epsilon > 0$, choose N so that $\|F_n - F_m\|_{(a,b)} < \epsilon$ if $n, m \ge N$ (Theorem 4.4.6); then $|L_n - L_m| < |L_n - F(x)| + |F_m(x) - L_m| + \epsilon$. Holding n and m fixed and letting $x \to x_0$ shows that $|L_n - L_m| \le \epsilon$ if $n, m \ge N$. Hence, $\lim_{n\to\infty} L_n = L$ exists (finite), by Theorem 4.1.13. Now choose n so that $|L - L_n| < \epsilon$ and $\|F_n - F\|_{(a,b)} < \epsilon$; then

$$\begin{aligned} |F(x) - L| &\le |F(x) - F_n(x)| + |F_n(x) - L_n| + |L_n - L| \\ &\le |F_n(x) - L_n| + 2\epsilon. \end{aligned}$$

For this fixed n there is a $\delta > 0$ such that $|F_n(x) - L_n| < \epsilon$ if $0 < |x - x_0| < \delta$. Therefore, $|F(x) - L| < 3\epsilon$ if $0 < |x - x_0| < \delta$; hence, $\lim_{x\to x_0} F(x) = L$.

4.4.14. **(a)** $F_n(x) = \dfrac{n}{x}\sin\dfrac{x}{n}$. From Taylor's theorem, $\sin\dfrac{x}{n} = \dfrac{x}{n} - \cos\theta(x,n)\dfrac{x^3}{6n^3}$ where $\theta(x,n)$ is between zero and $\dfrac{x}{n}$. Therefore, $|F_n(x) - 1| \le \dfrac{|x|}{6n^2}$, so $\|F_n - 1\|_{[1,4]} \le \dfrac{2}{3n^2} \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \displaystyle\int_1^4 F_n(x)\,dx = \int_1^4 1\,dx = 3$, by Theorem 4.4.9.

(b) $F_n(x) = \dfrac{1}{1 + x^{2n}}$; $\{F_n\}$ converges to $F(x) = \begin{cases} 1, & 0 \le x < 1, \\ \frac{1}{2}, & x = 1, \\ 0, & 1 < x < 2. \end{cases}$ Since $\|F_n\|_{[0,2]} = 1$ for $n \ge 1$ and F is integrable on $[0,2]$, $\lim_{n\to\infty} \displaystyle\int_0^2 F_n(x)\,dx = \int_0^2 F(x)\,dx = 1$, by Theorem 4.4.10.

(c) $\displaystyle\int_0^1 nxe^{-nx^2}\,dx = -\frac{1}{2}e^{-nx^2}\Big|_0^1 = \frac{1 - e^{-n}}{2} \to \frac{1}{2}$ as $n \to \infty$.

(d) $F_n(x) = \left(1 + \dfrac{x}{n}\right)^n$; since $\lim_{n\to\infty} F_n(x) = e^x$ and $\|F_n\|_{[0,1]} = \left(1 + \dfrac{1}{n}\right)^n \to e$ as $n \to \infty$, $\lim_{n\to\infty} \int_0^1 F_n(x)\,dx = \int_0^1 e^x\,dx = e - 1$, by Theorem 4.4.10.

4.4.15. $\left|\int_a^b F_n(x)\,dx - \int_a^b F_m(x)\,dx\right| \le (b-a)\|F_n - F_m\|_{[a,b]}$. Suppose that $\epsilon > 0$. Since $\{F_n\}$ converges uniformly on $[a,b]$ there is an integer N such that $\|F_n - F_m\| < \dfrac{\epsilon}{b-a}$ if $m, n \ge N$ (Theorem 4.4.2). Therefore, $\left|\int_a^b F_n(x)\,dx - \int_a^b F_m(x)\,dx\right| < \epsilon$ if $m, n \ge N$, so $\lim_{n\to\infty} \int_a^b F_n(x)\,dx$ exists (Theorem 4.1.13).

4.4.16. F and $F_1, F_2, \dots$ are nondecreasing (Exercise 4.4.2), so integrable (Theorem 3.2.9) on $[a,b]$. Now use Theorem 4.4.9.

4.4.17. **(a)** On $[-r/(1+r), r/(1-r)]$ if $0 < r < 1$; see Example 4.4.12 and let $M_n = n^{-1/2}r^n$. Therefore, Weierstrass's test implies that the series converges uniformly on compact subsets of $(-1/2, \infty)$.

(b) On $[-1/2, \infty)$; see Example 4.4.12, set $r = 1$, and let $M_n = n^{-3/2}$.

(c) Since $\sum nr^n < \infty$ if and only if $|r| < 1$, on any set S for which $\|x(1-x)\|_S \le r < 1$. Since $x(1-x) \le 1/4$ for all x, solving $x(1-x) = -1$ shows that S must be a closed subset of $\left(\dfrac{1-\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$.

(d) On $(-\infty, \infty)$; take $M_n = 1/n^2$.

(e) On $[r, \infty)$ with $r > 1$; take $M_n = n^{-r}$.

(f) Since $\sum r^n < \infty$ if and only if $|r| < 1$, on any set S such that $\|(1-x^2)/(1+x^2)\|_S \le r < 1$. Every x in such a set satisfies $[(1-r)/(1+r)]^{1/2} \le |x| \le [(1+r)/(1-r)]^{1/2}$. Compact subsets of $(-\infty, 0) \cup (0, \infty)$ have this property.

4.4.18. From Theorem 4.4.15 with $M_n = |a_n|$, the two series converge uniformly on $(-\infty, \infty)$. Theorem 4.4.18 implies that the sums are continuous on $(-\infty, \infty)$.

4.4.20. **(b)** Suppose that $\epsilon > 0$. Since $\sum |f_n|$ converges uniformly on S, there is an integer N such that $\||f_n| + |f_{n+1}| + \cdots + |f_m|\|_S < \epsilon$ if $m \ge n \ge N$ (Theorem 4.3.5). Therefore, $\|f_n + f_{n+1} + \cdots + f_m\|_S \le \||f_n\| + |f_{n+1}| + \cdots + |f_m|\|_S < \epsilon$ if $m \ge n \ge N$, so $\sum f_n$ converges uniformly on S, by Theorem 4.4.13

4.4.21. Suppose that $\epsilon > 0$. Since $\sum M_n < \infty$, there is an integer N such that $M_n + M_{n+1} + \cdots + M_m < \epsilon$ if $m \ge n \ge N$ (Theorem 4.3.5). Then $\||f_n| + |f_{n+1}| + \cdots + |f_m|\|_S \le \|f_n\|_S + \|f_{n+1}\|_S + \cdots + \|f_m\|_s \le M_n + M_{n+1} + \cdots + M_m < \epsilon$ if $m \ge n \ge N$, so $\sum |f_n|$ converges uniformly on S, by Theorem 4.4.13.

4.4.22. If $f_{n+1}(x) \le f_n(x)$, then $\sum_{j=k}^{N} |f_{n+1}(x) - f_n(x)| = \sum_{j=k}^{N} (f_n(x) - f_{n+1}(x)) = f_k(x) - f_{N+1}(x)$, so $\left\| \sum_{j=k}^{N} |f_{n+1} - f_n| - f_k \right\|_S \le \|f_{N+1}\|_S \to 0$ as $N \to \infty$. Now apply Theorem 4.4.16.

4.4.23. Apply Theorem 4.4.7 to the partial sums $F_n = \sum_{m=k}^{n} f_m(x)$.

4.4.24. Let $0 < \delta < \dfrac{\pi}{2}$ and let $I_m(\delta) = [2m\pi + \delta, 2(m+1)\pi - \delta]$, where m is an integer. From arguments like that in Example 4.3.21, the sequences $\{|\sin x + \cdots + \sin nx|\}$ and $\{|\cos x + \cdots + \cos nx|\}$ are bounded on $I_m(\delta)$. Therefore, Corollary 4.4.17 with $f_n = a_n$ implies that $\sum a_n \cos n\theta$ and $\sum a_n \sin n\theta$ converge uniformly on $I_m(\delta)$. Their sums are continuous functions of x on I_m (Theorem 4.4.7). If $2m\pi < x < 2(m+1)\pi$, then $x \in I_m(\delta)$ if δ is sufficiently small. Therfore the two sums are continuous at x.

4.4.26. Suppose that $\sum_{n=l}^{\infty} f_n$ converges pointwise to F and each f_n is integrable on $[a,b]$. Then:

(a) If the convergence is uniform, then F is integrable on $[a,b]$, and (A) $\int_a^b F(x)\,dx = \sum_{n=k}^{\infty} \int_a^b f_n(x)\,dx$.

(b) If the sequence $\left\{ \left\| \sum_{m=k}^{n} f_m \right\|_{[a,b]} \right\}_{n=k}^{\infty}$ is bounded and F is integrable on $[a,b]$, then (A) holds.

4.4.27. Theorem 4.4.19 justifies term by term integration in both parts.

(a) $e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \dfrac{t^{2n}}{n!}$; $\int_0^x e^{-t^2}\,dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n \dfrac{t^{2n}}{n!} \right) dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x \dfrac{t^{2n}}{n!}\,dt = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)n!}$.

(b) $\dfrac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \dfrac{t^{2n}}{(2n+1)!}$; $\int_0^x \dfrac{\sin t}{t}\,dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n \dfrac{t^{2n}}{(2n+1)!} \right) dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x \dfrac{t^{2n}}{(2n+1)!}\,dt = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)(2n+1)!}$.

4.4.28. Apply Theorem 4.4.11 to the partial sums $F_n = \sum_{m=k}^{n} f_m(x)$.

4.4.29. The series converges for $x = 0$ and the series obtained by differentiating it termwise converges uniformly on finite intervals (Example 4.4.17). Use Theorem 4.4.20.

4.4.30. The given series and those obtained by differentiating it term by term an even number of times obviously converge at $x_0 = 0$; those obtained by differentiating term by term an odd number of times converge at $x_0 = 0$, by the alternating series test. Since the series obtained by differentiating term by term k times converges uniformly on $(-\infty, \infty)$ by Weierstrass's test, the conclusion follows by repeated application of Theorem 4.4.20.

4.4.31. The graph of $y = f_n(x)$ is a triangle with height n and width $2/n^3$; hence, the area under the graph is $1/n^2$. Since $\sum 1/n^2 < \infty$ and F is continuous, the conclusion follows.

4.5 POWER SERIES

4.5.1. $\sum a_n R^n$ and $\sum(-1)^n a_n R^n$ both converge absolutely if and only if $\sum |a_n| R^n < \infty$.

4.5.2. **(a)** $|a_n|^{1/n} = \left(1 + \frac{1}{n}\right)^{1/n} (2 + (-1)^n)$; $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 3e$; $R = 1/3e$.

(b) $|a_n|^{1/n} = 2^{1/\sqrt{n}} \to 1$ as $n \to \infty$; $R = 1$.

(c) $|a_n|^{1/n} = 2 + \sin \frac{n\pi}{6}$; $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 3$; $R = 1/3$.

(d) $|a_n|^{1/n} = n^{1/\sqrt{n}} \to 1$ as $n \to \infty$; $R = 1$.

(e) $|a_n|^{1/n} = \frac{1}{n} \to 0$ as $n \to \infty$; $R = \infty$.

4.5.3. **(a)** If $|a_n r^n| \le M$, then $|a_n(x_1 - x_0)^n| \le M\rho^n$, where $\rho = |x_1 - x_0|/r < 1$, and $\sum |a_n(x_1 - x_0)^n| < \infty$ because $\sum \rho^n < \infty$.

(b) If $\{a_n(x_1 - x_0)^n\}$ is bounded, then $\sum a_n(x - x_0)^n$ converges if $|x - x_0| < |x_1 - x_0|$ (from **(a)** with $r = |x_1 - x_0|$). This is a contradiction if $|x_1 - x_0| > R$.

4.5.4. $\lim |g(n)|^{1/n} = 1$ if g is rational; hence, $\overline{\lim}_{n\to\infty} |a_n g(n)|^{1/n} = \overline{\lim}_{n\to\infty} |a_n|^{1/n}$ (Exercise 4.1.30**(a)**).

4.5.5. Since $1/R_1 > 1/R$, there is an integer k such that $|a_n|^{1/n} \le 1/R_1$ if $n > k$. Therefore, $|a_n||x - x_0|^n \le (r/R_1)^n$ if $|x - x_0| < r$ and $n > k$, so

$$\left| f(x) - \sum_{n=0}^{k} a_n(x - x_0)^n \right| \le \sum_{n=k+1}^{\infty} |a_n||x - x_0|^n \le \sum_{n=k+1}^{\infty} \left(\frac{r}{R_1}\right)^n = \left(\frac{r}{R_1}\right)^{k+1} \frac{R_1}{R_1 - r}.$$

4.5.6. The series $g(x)$ converges if $|x^k| < R$ and diverges if $|x^k| > R$. This implies the result.

4.5.7. **(a)** If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = \infty$, then $\lim_{n\to\infty} \frac{|a_{n+1}(z - z_0)^{n+1}|}{|a_n(z - z_0)^n|} = \infty$ for any $z \ne z_0$; hence, $\sum |a_n(z - z_0)^n| = \infty$ if $z \ne z_0$ (Theorem 4.3.14**(b)**), and $R = 0$.

(b) If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 0$, then $\lim_{n\to\infty} \frac{|a_{n+1}(z - z_0)^{n+1}|}{|a_n(z - z_0)^n|} = 0$ for any $z \ne z_0$; hence, $\sum |a_n(z - z_0)^n| < \infty$ if $z \ne z_0$ (Theorem 4.3.14**(a)**), and $R = \infty$.

4.5.8. We use Theorem 4.5.3 in all parts.

(a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\log(n+1)}{\log n} = \lim_{n\to\infty}\frac{n}{n+1}$ (L'Hospital's rule) $= 1$; $R = 1$.

(b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}(n+1)^p}{2^n n^p} = 2\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^p = 2$; $R = \frac{1}{2}$.

(c) $\left|\frac{a_{n+1}}{a_n}\right| = \binom{2n+2}{n+1}\Big/\binom{2n}{n} = \frac{(2n+2)!}{(n+1)!(n+1)!}\frac{n!n!}{(2n)!} = 2\left(\frac{2n+1}{n+1}\right) \to 4$ as $n\to\infty$; $R = \frac{1}{4}$.

(d) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2+1}{(n+1)4^{n+1}}\frac{n4^n}{n^2+1} = \frac{n}{4(n+1)}\frac{(n+1)^2+1}{n^2+1} \to \frac{1}{4}$ as $n\to\infty$; $R = 4$.

(e) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \left(1+\frac{1}{n}\right)^n \to e$ as $n\to\infty$; $R = \frac{1}{e}$.

(f) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\alpha+n}{\beta+n} \to 1$ as $n\to\infty$; $R = 1$.

4.5.9. **(a)** Let $\underline{L} = \varliminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. The conclusion is obvious if $\underline{L} = 0$. If $0 < \epsilon < \underline{L}$ there is an integer N such that $\left|\frac{a_{m+1}}{a_m}\right| > \underline{L} - \epsilon$ if $m \ge N$. Therefore, $|a_n| > (\underline{L}-\epsilon)^{n-N}|a_N|$, so $|a_n|^{1/n} > \left[|a_n|(\underline{L}-\epsilon)^{-N}\right]^{1/n}(\underline{L}-\epsilon)$ if $n \ge N$. Hence, $\varliminf_{n\to\infty}|a_n|^{1/n} \ge \underline{L}-\epsilon$. Since this holds for every $\epsilon > 0$, $\varliminf_{n\to\infty}|a_n|^{1/n} \ge \underline{L}$.

(b) Let $\overline{L} = \varlimsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. The conclusion is obvious if $\overline{L} = \infty$. If $\overline{L} < \infty$ and $\epsilon > 0$, there is an integer N such that $\left|\frac{a_{m+1}}{a_m}\right| < \overline{L} + \epsilon$ if $m \ge N$. Therefore, $|a_n| < (\overline{L} + \epsilon)^{n-N}|a_N|$, so $|a_n|^{1/n} < \left[|a_n|(\overline{L}+\epsilon)^{-N}\right]^{1/n}(\overline{L}+\epsilon)$ if $n \ge N$. Hence, $\varlimsup_{n\to\infty}|a_n|^{1/n} \le \overline{L}+\epsilon$. Since this holds for every $\epsilon > 0$, $\varlimsup_{n\to\infty}|a_n|^{1/n} \le \overline{L}$.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, then $\underline{L} = \overline{L} = L$, so $\varliminf_{n\to\infty}|a_n|^{1/n} = L$ and therefore $\varlimsup_{n\to\infty}|a_n|^{1/n} = L$. Hence, $R = \frac{1}{L}$, by Theorem 4.5.2.

4.5.10. Differentiating and multiplying by x yields $\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$, $|x| < 1$. Repeating this yields $\frac{x(1+x)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2x^n$, $|x| < 1$.

4.5.11. We apply Theorem 4.5.5 in all parts.

(a) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$;

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \left(\frac{x}{2}\right)^{2n-1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} = -J_1(x).$$

(b)

$$\begin{aligned}
J_p'(x) &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(2n+p)}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p-1} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n(n+p)}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p-1} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p-1} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p-1)!} \left(\frac{x}{2}\right)^{2n+p-1} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!(n+p)!} \left(\frac{x}{2}\right)^{2n+p-1} \\
&= \frac{1}{2}J_{p-1}(x) - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p+1)!} \left(\frac{x}{2}\right)^{2n+p+1} = \frac{1}{2}(J_{p-1}(x) - J_{p+1}(x)).
\end{aligned}$$

(c)

$$\begin{aligned}
x^2 J_p'' + xJ_p' + (x^2 - p^2)J_p = &\sum_{n=0}^{\infty} \frac{(-1)^n(2n+p)(2n+p-1)}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p} \\
&+ \sum_{n=0}^{\infty} \frac{(-1)^n(2n+p)}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p} \\
&+4\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p+2} \\
&-p^2\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p}.
\end{aligned}$$

After rewriting the third sum as

$$4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(n+p-1)!} \left(\frac{x}{2}\right)^{2n+p} = -4\sum_{n=1}^{\infty}(-1)^n \frac{n(n+p)}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p},$$

we obtain $x^2J_p''+xJ_p'+(x^2-p^2)J_p = \sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^{2n+p}$, with $a_0 = \frac{p(p-1)+p-p^2}{p!} =$ 0 and $a_n = \frac{(2n+p)(2n+p-1)+(2n+p)-4n(n+p)-p^2}{n!(n+p)!} = 0.$

4.5.12. $\sum_{n=1}^{\infty} na_n x^{n-1} = -2\sum_{n=0}^{\infty} a_n x^{n+1} = -2\sum_{n=2}^{\infty} a_{n-2}x^{n-1}$; since $f(0) = 1$, $a_0 = 1$. From Corollary 4.5.7, the first and last power series are identical on an interval containing $x_0 = 0$ if and only if $a_1 = 0$ and $a_n = -\dfrac{2a_{n-2}}{n}$, $n \geq 2$. Therefore, $a_{2m-1} = -\dfrac{2a_{2m-1}}{2m-1}$ and $a_{2m} = -\dfrac{a_{2m-2}}{m}$, $m \geq 0$. Since $a_0 = 1$ and $a_1 = 0$, $a_{2m-1} = 0$ and $a_{2m} = \dfrac{(-1)^m}{m!}$ for $m \geq 1$, by induction, so $f(x) = \sum_{m=0}^{\infty} \dfrac{(-1)^m}{m!} x^{2m} = e^{-x^2}$.

4.5.13. For $|x| < R^{1/k}$, $g(x) = \sum_{n=0}^{\infty} a_n x^{kn}$, which can also be written as $g(x) = \sum_{m=0}^{\infty} b_m x^m$ with $b_m = \begin{cases} 0 & \text{if } m \neq kn, \\ a_n & \text{if } m = kn \end{cases}$ (k = nonnegative integer) (Corollary 4.5.7). Since $b_m = \dfrac{g^{(m)}(0)}{m!}$ and $a_n = \dfrac{f^n(0)}{n!}$ (Corollary 4.5.6), the conclusion now follows.

4.5.14. Repeated application of Rolle's theorem shows that there are sequences $\{t_{nk}\}$ such that $f^{(k)}(t_{nk}) = 0$ and $\lim_{n\to\infty} t_{nk} = x_0$, $k \geq 1$. Since $f^{(k)}$ is continuous at x_0, $f^{(k)}(x_0) = 0$ and therefore $a_k = 0$, $k \geq 0$, by Corollary 4.5.6.

4.5.15. From Theorem 4.5.2, $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges uniformy to f on $[x_1, x_2]$. Now use Theorem 4.4.19.

4.5.16. $\dfrac{1}{x} = \dfrac{1}{1 + (x-1)} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$, $|x-1| < 1$. Therefore, $\log x = \int_1^x \dfrac{dt}{t} = \sum_{n=0}^{\infty}(-1)^n \int_1^x (t-1)^n\, dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1}(x-1)^{n+1}$, $|x - 1| < 1$, by Theorem 4.5.8. Hence, $\dfrac{\log x}{x-1} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1}(x-1)^n$, so $\int_1^x \dfrac{\log t}{t-1}\, dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} \int_1^x (t-1)^n\, dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(n+1)^2}(x-1)^{n+1} = \sum_{n=0}^{\infty} \dfrac{(-1)^{n-1}}{n}(x-1)^n$, $|x-1| < 1$, again by Theorem 4.5.8; $R = 1$.

4.5.17. $\operatorname{Tan}^{-1} x = \int_0^x \dfrac{dt}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$, $|x < 1|$, by Theorem 4.5.8. From Corollary 4.5.6, $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n \dfrac{(2n+1)!}{2n+1} = (-1)^n(2n)!$; $\dfrac{\pi}{6} = \operatorname{Tan}^{-1}\dfrac{1}{\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)3^{n+1/2}}$.

4.5.18. If $F'(x) = f(x)$, then integrating term by term (Theorem 4.5.8) yields the result.

4.5.19. Use Theorem 4.5.8 repeatedly to show that f can be represented by a power series in $(x - x_0)$ in the interval. Then use Theorem 4.5.5.

4.5.20. Since the identity holds with $x = 0$, it suffices to verify that the derivative of the left side is zero if $|x| < 1$. The derivative is

$$d(x) = -q(1+x)^{-q-1}\sum_{n=0}^{\infty}\binom{q}{n}x^n + (1+x)^{-q}\sum_{n=1}^{\infty} n\binom{q}{n}x^{n-1}.$$

Since $n\binom{q}{n} = q\binom{q-1}{n-1}$ and $\displaystyle\sum_{n=1}^{\infty}\binom{q-1}{n-1}x^{n-1} = \sum_{n=0}^{\infty}\binom{q-1}{n}x^n = (1+x)^{q-1}$, $d(x) = -q(1+x)^{-q-1}(1+x)^q + q(1+x)^{-q}(1+x)^{q-1} = 0$, $|x| < 1$.

4.5.21. For a fixed x such that $|x - x_0| < \min R_1, R_2$, the series $\displaystyle f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\displaystyle g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ both converge, and Theorem 4.3.10 implies that $\displaystyle \alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty}(a\alpha + b\beta)(x-x_0)^n$.

4.5.22. If $f(x) = \cosh x$, then $f^{(2m)}(x) = \cosh x$ and $f^{(2m+1)}(x) = \sinh x$, so $f^{(2m)}(0) = 1$, $f^{(2m+1)}(0) = 0$, and $\displaystyle f(x) = \sum_{m=0}^{\infty}\frac{x^{2m}}{(2m)!}$.

If $f(x) = \sinh x$, then $f^{(2m)}(x) = \sinh x$ and $f^{(2m+1)}(x) = \cosh x$, so $f^{(2m)}(0) = 0$, $f^{(2m+1)}(0) = 1$, and $\displaystyle f(x) = \sum_{m=0}^{\infty}\frac{x^{2m+1}}{(2m+1)!}$.

Since $\displaystyle e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $\displaystyle e^{-x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, $\displaystyle \cosh x = \frac{e^x + e^{-x}}{2} = \sum_{m=0}^{\infty}\frac{x^{2m}}{(2m)!}$ and $\displaystyle \sinh x = \frac{e^x - e^{-x}}{2} = \sum_{m=0}^{\infty}\frac{x^{2m+1}}{(2m+1)!}$.

4.5.24. **(a)**
$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) \\ &= x + x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(\frac{1}{120} - \frac{1}{12}\right)x^5 + \cdots \\ &= x + x^2 + \frac{x^3}{3} - \frac{3x^5}{40} + \cdots. \end{aligned}$$

(b)
$$\begin{aligned} \frac{e^{-x}}{1+x^2} &= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)(1 - x^2 + \cdots) \\ &= 1 - x + \left(\frac{1}{2} - 1\right)x^2 + \left(1 - \frac{1}{6}\right)x^3 + \cdots \\ &= 1 - x - \frac{x^2}{2} + \frac{5x^3}{6} + \cdots. \end{aligned}$$

(c)
$$\begin{aligned} \frac{\cos x}{1+x^6} &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)(1 - x^6 \cdots) \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{721x^6}{720} + \cdots. \end{aligned}$$

(d)

$$\begin{aligned}(\sin x)\log(1+x) &= \left(x-\frac{x^3}{6}+\frac{x^5}{120}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\cdots\right)\\ &= x^2-\frac{x^3}{2}+\left(\frac{1}{3}-\frac{1}{6}\right)x^4+\left(-\frac{1}{4}+\frac{1}{12}\right)x^5+\cdots\\ &= x^2-\frac{x^3}{2}+\frac{x^4}{6}-\frac{x^5}{6}+\cdots.\end{aligned}$$

4.5.25. $2\sin x\cos x = 2\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\right) = \sum_{n=0}^{\infty}c_nx^{2n+1}$, where $c_n = 2\sum_{r=0}^{n}(-1)^r(-1)^{n-r}\frac{1}{(2r)!(2n-2r+1)!} = (-1)^n\frac{2}{(2n+1)!}\sum_{r=0}^{n}\binom{2n+1}{2r}$. Adding $2^{2n+1} = (1+1)^{2n+1} = \sum_{r=0}^{2n+1}\binom{2n+1}{r}$ and $0 = (1-1)^{2n+1} = \sum_{r=0}^{2n+1}(-1)^r\binom{2n+1}{r}$ yields $2\sum_{r=0}^{n}\binom{2n+1}{2r} = 2^{2n+1}$, so $2\sin x\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!} = \sin 2x$.

4.5.26. **(a)** Differentiating (A) with respect to x yields

$$(1-2xt+x^2)^{-3/2}(t-x) = \sum_{n=0}^{\infty}(n+1)P_{n+1}(t)x^n;$$

hence,

$$(t-x)\sum_{n=0}^{\infty}P_n(t)x^n = (1-2xt+x^2)\sum_{n=0}^{\infty}(n+1)P_{n+1}(t)x^n.$$

Performing the indicated multiplications and shifting indices yields

$$\begin{aligned}tP_0(t)+\sum_{n=1}^{\infty}\left[tP_n(t)-P_{n-1}(t)\right]x^n &= P_1(t)+\left[2P_2(t)-2tP_1(t)\right]x\\ &\quad+\sum_{n=2}^{\infty}[(n+1)P_{n+1}(t)-2ntP_n(t)+(n-1)P_{n-1}(t)]x^n.\end{aligned}$$

Equating like powers of x yields $P_1 = tP_0(t)$,

$$P_2 = \frac{3tP_1-P_0}{2} \quad\text{and}\quad P_{n+1} = \frac{2n+1}{n+1}tP_n-\frac{n}{n+1}P_{n-1}, \quad n\ge 2.$$

Setting $x=0$ in (A) shows that $P_0(t)=1$. Therefore, $P_1(t)=tP_0(t)=t$.

(b) Proof by induction: P_0 and P_1 are polynomials of degree 0 and 1 respectively. Now suppose that $n\ge 2$ and $P_0, P_1, \dots, P_n$ are polynomials of degrees $0, 1, \dots, n$ respectively. Then the recursion formula implies that $\deg(P) = n+1$.

4.5.27. **(a)** $e^x = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\dfrac{\sin x}{x}$;

$$\begin{aligned} 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\frac{\sin x}{x} \\ &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)\left(1 - \frac{x^2}{6} + \cdots\right) \\ &= a_0 + a_1 x + \left(a_2 - \frac{a_0}{6}\right)x^2 + \left(a_3 - \frac{a_1}{6}\right)x^3 + \cdots; \end{aligned}$$

$a_0 = 1, a_1 = 1, a_2 = \dfrac{1}{2} + \dfrac{a_0}{6} = \dfrac{2}{3}, a_3 = \dfrac{1}{6} + \dfrac{a_1}{6} = \dfrac{1}{3}$.

(b) $\cos x = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(1 + x + x^2)$;

$$\begin{aligned} 1 - \frac{x^2}{2} + \cdots &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(1 + x + x^2) \\ &= a_0 + (a_1 + a_0)x + (a_0 + a_1 + a_2)x^2 + (a_1 + a_2 + a_3)x^3 + \cdots; \end{aligned}$$

$a_0 = 1$; $a_0 + a_1 = 0$; $a_0 + a_1 + a_2 = -\dfrac{1}{2}$; $a_1 + a_2 + a_3 = 0$; $a_0 = 1$, $a_0 = -1$, $a_2 = -\dfrac{1}{2}$, $a_3 = \dfrac{3}{2}$.

(c) $\sec x = a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + \cdots$;

$$\begin{aligned} 1 &= (a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + \cdots)\cos x \\ &= (a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + \cdots)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) \\ &= a_0 + \left(a_1 - \frac{1}{2}\right)x^2 + \left(a_2 - \frac{a_1}{2} + \frac{a_0}{24}\right)x^4 + \left(a_3 - \frac{a_2}{2} + \frac{a_1}{24} - \frac{a_0}{720}\right)x^6 + \cdots; \end{aligned}$$

$a_0 = 1$; $a_1 = \dfrac{1}{2}$; $a_2 = \dfrac{a_1}{2} - \dfrac{a_0}{24} = \dfrac{5}{24}$; $a_3 = \dfrac{a_2}{2} - \dfrac{a_1}{24} + \dfrac{a_0}{720} = \dfrac{61}{720}$.

(d) $x \csc x = a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + \cdots$;

$$\begin{aligned} 1 &= (a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + \cdots)\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots\right) \\ &= a_0 + \left(a_1 - \frac{a_0}{6}\right)x^2 + \left(a_2 - \frac{a_1}{6} + \frac{a_0}{120}\right)x^4 + \left(a_3 - \frac{a_2}{6} + \frac{a_1}{120} - \frac{a_0}{5040}\right)x^6 + \cdots; \end{aligned}$$

$a_0 = 1$; $a_1 = \dfrac{a_0}{6} = \dfrac{1}{6}$; $a_2 = \dfrac{a_1}{6} - \dfrac{a_0}{120} = \dfrac{7}{360}$; $a_3 = \dfrac{a_2}{6} - \dfrac{a_1}{120} + \dfrac{a_0}{5040} = \dfrac{31}{15120}$.

(f)

$$\begin{aligned} 2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120} - \frac{(2x)^7}{5040} + \cdots &= x(a_0 + a_1x^2 + a_2x^4 + a_3x^6 + \cdots) \\ &\quad \times \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots\right) \\ &= a_0x + \left(a_1 - \frac{a_0}{6}\right)x^3 + \left(a_2 - \frac{a_1}{6} + \frac{a_0}{120}\right)x^5 \\ &\quad + \left(a_3 - \frac{a_2}{6} + \frac{a_1}{120} - \frac{a_0}{5040}\right)x^7 + \cdots ; \end{aligned}$$

$a_0 = 2; -\frac{4}{3} = a_1 - \frac{a_0}{6}; a_1 = \frac{a_0}{6} - \frac{4}{3} = -1; \frac{4}{15} = a_2 - \frac{a_1}{6} + \frac{a_0}{120}; a_2 = \frac{4}{15} + \frac{a_1}{6} - \frac{a_0}{120} = \frac{1}{12}; -\frac{8}{315} = a_3 - \frac{a_2}{6} + \frac{a_1}{120} - \frac{a_0}{5040}; a_3 = -\frac{8}{315} + \frac{a_2}{6} - \frac{a_1}{120} + \frac{a_0}{5040} = -\frac{1}{360}.$

4.5.28. **(a)** Multiplying both sides of the recurrence formula by x^{n+1} and summing over $n \geq 1$ yields $\sum_{n=1}^{\infty} a_{n+1}x^{n+1} = \sum_{n=1}^{\infty} a_n x^{n+1} + 6\sum_{n=1}^{\infty} a_{n-1}x^{n+1}$; hence, $F(x) - 5 - 5x = x\,[F(x) - 5] + 6x^2F(x)$. Solving this for $F(x)$ yields $F(x) = \frac{5}{(1-3x)(1+2x)}$.

(b) Write $F(x) = \frac{A}{1-3x} + \frac{B}{1+2x}$, where $A(1+2x) + B(1-3x) = 5$. Setting $x = -\frac{1}{2}$ yields $B = 2$ and setting $x = \frac{1}{3}$ yields $A = 3$, so $F(x) = \frac{3}{1-3x} + \frac{2}{1+2x}$, and expanding the two terms as geometric series yields $F(x) = \sum_{n=0}^{\infty}[3^{n+1} - (-2)^{n+1}]x^n$.

4.5.29. From the given expansion and Theorem 4.5.8, $(x-1)\log(1-x) - x = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(n+2)}$, $|x| < 1$. Since $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} < \infty$, Abel's theorem implies that $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = -\lim_{x\to 0-}((x-1)\log(1-x) - x) = 1$.

4.5.30. The series converges by the alternating series test if $-1 < q < 0$, since (A) $\binom{q}{n}\Big/\binom{q}{n+1} = \frac{q-n}{n+1}$ and $\lim_{n\to\infty}\binom{q}{n} = 0$ (Exercise 4.1.35). Since $\sum_{n=0}^{\infty}\binom{q}{n}x^n = (1+x)^q$ if $|x| < 1$, the stated identity also holds for $-1 < q < 0$, by Abel's theorem. If $q \leq -1$, then (A) implies that the series diverges.

4.5.31. **(a)** Let $s_m = \sum_{k=n}^{m} b_k$. Summation by parts yields

$$\left|\sum_{k=n}^{m} b_k x^k\right| = \left|(1-x)\sum_{k=n}^{m-1} s_k x^k + s_m x^m\right| \leq (1-x)\epsilon \sum_{k=n}^{m-1} x^k + \epsilon x^m \leq 2\epsilon$$

if $0 \le x \le 1$, $m \ge n \ge N$. Use Theorem 4.4.13.

(b) If $\sum_{n=0}^{\infty} b_n$ converges, then $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converges uniformly on $[0, 1]$ **(a)**, so g is continuous on $[0, 1]$ (Theorem 4.4.18); hence, $\lim_{x\to 1-} g(x) = g(1)$.

4.5.32. From Exercise 4.5.31, the three series in the identity $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \left(\sum_{n=0}^{\infty} c_n x^n\right)$ converge uniformly on $[0, 1]$, and are therefore continuous on $[0, 1]$. Letting $x \to 1-$ yields the result.

4.5.33. If $\sum_{n=0}^{\infty} b_n < \infty$, use Theorem 4.5.12. If $\sum b_n = \infty$ and M is arbitrary, then $\sum_{n=0}^{k} b_n > M$ for some k. Hence, there is a δ such that $0 < \delta < 1$ and $\sum_{n=0}^{k} b_n x^n > M$ if $1 - \delta < x < 1$. Since $b_n \ge 0$ for all n, this implies that $g(x) > M$ if $1 - \delta < x < 1$. Hence, $\lim_{x\to 1-} g(x) = \infty$.

4.5.34. Integrating the binomial series $(1 - x^2)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n}$, $|x| < 1$, from 0 to x yields $\sin^{-1} x = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1}$, $|x| < 1$. But

$$(-1)^n \binom{-\frac{1}{2}}{n} = \frac{(2n-1)(2n-3)\dots(1)}{2^n n!} = \frac{1}{2^{2n}}\binom{2n}{n},$$

so $\sin^{-1}(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{2n+1}}{2^{2n}(2n+1)}$, $|x| < 1$. Let $x \to 1-$ and use Exercise 4.5.33 to obtain the result.

CHAPTER 5
Real-Valued Functions of Several Variables

5.1 STRUCTURE OF $\mathbb{R}^n$

5.1.1. **(a)** $3\mathbf{X} + 6\mathbf{Y} = 3(1, 2-3, 1) + 6(0, -1, 2, 0) = (3, 6, -9, 3) + (0, -6, 12, 0) = (3, 0, 3, 0)$.

(b) $(-1)\mathbf{X} + 2\mathbf{Y} = (-1)(1, -1, 2) + 2(0, -1, 3) = (-1, 1, -2) + (0, -2, 6) = (-1, -1, 4)$.

(c) $\frac{1}{2}\mathbf{X} + \frac{1}{6}\mathbf{Y} = \frac{1}{2}(\frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{1}{6}) + \frac{1}{6}(-\frac{1}{2}, 1, 5, \frac{1}{3}) = (\frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{1}{12}) + (-\frac{1}{12}, \frac{1}{6}, \frac{5}{6}, \frac{1}{18}) = (\frac{1}{6}, \frac{11}{12}, \frac{23}{24}, \frac{5}{36})$.

5.1.2. **(a)** $\mathbf{X} + \mathbf{Y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n) = \mathbf{Y} + \mathbf{X}$.

(b) $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \dots, (x_n + y_n) + z_n) = (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \dots, x_n + (y_n + z_n)) = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$.

(c) If $\mathbf{0} = (0, 0, \dots, 0)$, then $\mathbf{0} + \mathbf{X} = (0 + x_1, 0 + x_2, \dots, 0 + x_n) = (x_1, x_2, \dots, x_n) = \mathbf{X}$ for every $\mathbf{X}$. Now suppose that $\mathbf{Y}$ is a vector such that $\mathbf{Y} + \mathbf{X} = \mathbf{X}$ for some $\mathbf{X}$. Then $(y_1 + x_1, y_2 + x_2, \dots, y_n + x_n) = (x_1, x_2, \dots, x_n)$; hence, $y_i + x_i = x_i$ and therefore $y_i = 0$, $1 \le i \le n$, so $\mathbf{Y} = \mathbf{0}$.

(d) $\mathbf{X} + (-\mathbf{X}) = (x_1, x_2, \dots, x_n) + (-x_1, -x_2, \dots, -x_n) = (x_1 - x_1, x_2 - x_2, \dots, x_n - x_n) = (0, 0, \dots, 0) = \mathbf{0}$. If $\mathbf{X} + \mathbf{Y} = \mathbf{0}$, then $(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (0, 0, \dots, 0)$; hence, $x_i + y_i = 0$ and therefore $y_i = -x_i$, $1 \le i \le n$, so $\mathbf{Y} = -\mathbf{X}$.

(e) $a(b\mathbf{X}) = a(bx_1, bx_2, \dots, bx_n) = (a(bx_1), a(bx_2), \dots, a(bx_n)) = ((ab)x_1, (ab)x_2, \dots, (ab)x_n) = (ab)\mathbf{X}$.

(f) $(a+b)\mathbf{X} = ((a+b)x_1, (a+b)x_2, \dots, (a+b)x_n) = (ax_1 + bx_1, ax_2 + bx_2, \dots, ax_n + bx_n) = (ax_1, ax_2, \dots, ax_n) + (bx_1, bx_2, \dots, bx_n) = a\mathbf{X} + b\mathbf{X}$

(g) $a(\mathbf{X} + \mathbf{Y}) = a(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (a(x_1 + y_1), a(x_2 + y_2), \dots, a(x_n + y_n)) = (ax_1 + ay_1, ax_2 + ay_2, \dots, ax_n + ay_n) = (ax_1, ax_2, \dots, ax_n) + (ay_1, ay_2, \dots, ay_n) = a\mathbf{X} + a\mathbf{Y}$.

(h) $1\mathbf{X} = (1x_1, 1x_2, \dots, 1x_n) = (x_1, x_2, \dots, x_n) = \mathbf{X}$.

5.1.3. **(a)** $|\mathbf{X}| = \left(1^2 + 2^2 + (-3)^2 + 1^2\right)^{1/2} = \sqrt{15}$.

(b) $|\mathbf{X}| = \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{6^2}\right)^{1/2} = \frac{\sqrt{65}}{12}$.

(c) $|\mathbf{X}| = \left(1^2 + 2^2 + (-1)^2 + 3^2 + 4^2\right)^{1/2} = \sqrt{31}$.

(d) $|\mathbf{X}| = \left(0^2 + 1^2 + 0^2 + (-1)^2 + 0^2 + (-1)^2\right)^{1/2} = \sqrt{3}$.

5.1.4. **(a)** $|\mathbf{X} - \mathbf{Y}| = \left((3-2)^2 + (4-0)^2 + (5+1)^2 + (-4-2)^2\right)^{1/2} = \sqrt{89}$.

(b) $|\mathbf{X} - \mathbf{Y}| = \left(\left(-\frac{1}{2} - \frac{1}{3}\right)^2 + \left(\frac{1}{2} + \frac{1}{6}\right)^2 + \left(\frac{1}{4} - \frac{1}{6}\right)^2 + \left(-\frac{1}{4} + \frac{1}{3}\right)^2\right)^{1/2} = \frac{\sqrt{166}}{12}$.

(c) $|\mathbf{X} - \mathbf{Y}| = \left((0-2)^2 + (0+1)^2 + (0-2)^2\right)^{1/2} = 3$.

(d) $|\mathbf{X} - \mathbf{Y}| = \left((3-2)^2 + (-1-0)^2 + (4-1)^2 + (0+4)^2 + (-1-1)^2\right)^{1/2} = \sqrt{31}$.

5.1.5. **(a)** $\mathbf{X} \cdot \mathbf{Y} = 3(3) + 4(0) + 5(3) + (-4)3 = 12$.

(b) $\mathbf{X} \cdot \mathbf{Y} = \frac{1}{6}\left(-\frac{1}{2}\right) + \frac{11}{12}\left(\frac{1}{2}\right) + \frac{9}{8}\left(\frac{1}{4}\right) + \frac{5}{2}\left(-\frac{1}{4}\right) = \frac{1}{32}$.

(c) $\mathbf{X} \cdot \mathbf{Y} = 1(1) + 2(2) + (-3)(-1) + 1(3) + 4(4) = 27$.

5.1.6. **(a)** $|a\mathbf{X}| = (a^2x_1^2 + a^2x_2^2 + \cdots + a^2x_n^2)^{1/2} = |a|(x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} = |a||\mathbf{X}|$.

(b) $|\mathbf{X}| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \ge 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n = 0$; that is, if and only if $\mathbf{X} = \mathbf{0}$.

(c) $|\mathbf{X} - \mathbf{Y}| = ((x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2)^{1/2} \ge 0$, with equality if and only if $x_1 - y_1 = x_2 - y_2 = \cdots = x_n - y_n = 0$; that is, if and only if $\mathbf{X} = \mathbf{Y}$.

(d) $\mathbf{X} \cdot \mathbf{Y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = y_1x_1 + y_2x_2 + \cdots + y_nx_n = \mathbf{Y} \cdot \mathbf{X}$.

(e) $\mathbf{X} \cdot (\mathbf{Y} + \mathbf{Z}) = x_1(y_1 + z_1) + x_2(y_2 + z_2) + \cdots + x_n(y_n + z_n) = x_1y_1 + x_2y_2 + \cdots x_ny_n + x_1z_1 + x_2z_2 + \cdots + x_nz_n = \mathbf{X} \cdot \mathbf{Y} + \mathbf{X} \cdot \mathbf{Z}$.

(f) $(c\mathbf{X}) \cdot \mathbf{Y} = (cx_1)y_1 + (cx_2)y_2 + \cdots + (cx_n)y_n = x_1(cy_1) + x_2(cy_2) + \cdots + x_n(cy_n) = \mathbf{X} \cdot (c\mathbf{Y}) = c(x_1y_1) + c(x_2y_2) + \cdots + c(x_ny_n) = c(\mathbf{X} \cdot \mathbf{Y})$.

5.1.8. If both equations represent the same line, then $\mathbf{X}_0 = \mathbf{X}_1 + s_0\mathbf{V}$ and $\mathbf{X}_0 + \mathbf{U} = \mathbf{X}_1 + s_1\mathbf{V}$ for some s_0 and s_1; that is, $\mathbf{X}_1 - \mathbf{X}_0 = s_0\mathbf{V}$ and $\mathbf{U} = (s_1 - s_0)\mathbf{V}$ are multiples of $\mathbf{V}$. Conversely, if $\mathbf{X}_1 - \mathbf{X}_0 = a\mathbf{V}$ and $\mathbf{U} = b\mathbf{V}$, then $\mathbf{X}_0 + t\mathbf{U} = \mathbf{X}_1 + (a + bt)\mathbf{V}$ and since there is for each s a unique t such that $s = a + bt$, the two equations represent the same line.

5.1.9. In all cases $\mathbf{X} = \mathbf{X}_0 + t(\mathbf{X}_1 - \mathbf{X}_0)$.

5.1.10. In all cases let $\rho = \sup\left\{\epsilon \,\middle|\, N_\epsilon(\mathbf{X}_0) \subset S\right\}$.

(a) If S is a sphere with center $\mathbf{X}_1$ and radius r and $\mathbf{X}_0 \in S$, then $\rho = r - |\mathbf{X}_0 - \mathbf{X}_1|$; in this case $\mathbf{X}_1 = (0, 3, -2, 2)$, $r = 7$, and $\mathbf{X}_0 = (1, 2, -1, 3)$. Since $|\mathbf{X}_0 - \mathbf{X}_1| = |(1, -1, 1, 1)| = 2$, $\rho = 5$.

(b) If $S = \left\{(x_1, x_2, \dots, x_n) \,\middle|\, |x_i| \le r_i,\ 1 \le i \le n\right\}$ and $\mathbf{X}_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}) \in S$, then $\rho = \min_{1\le i\le n} \min\{r_i + x_i^{(0)}, r_i - x_i^{(0)}\}$. In this case $\mathbf{X_0} = (1, 2, -1, 3)$ and $r_i = 5, i = 1$, 2, 3, 4, so $\rho = \min\{6, 4, 7, 3, 4, 6, 8, 2\} = 2$.

(c) If $\mathbf{X}_0 \in \mathbb{R}^2$ is in a triangle S, then ρ is the smallest of the distances from $\mathbf{X_0}$ to lines parallel to the three sides of S. If the vertices of S are $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, then these three distances are given by

$$d_1 = \frac{\sqrt{|\mathbf{X}_0 - \mathbf{X}_3|^2|\mathbf{X}_2 - \mathbf{X}_3|^2 - |(\mathbf{X}_0 - \mathbf{X}_3)\cdot(\mathbf{X}_2 - \mathbf{X}_3)|^2}}{|\mathbf{X}_2 - \mathbf{X}_3|},$$
$$d_2 = \frac{\sqrt{|\mathbf{X}_0 - \mathbf{X}_1|^2|\mathbf{X}_3 - \mathbf{X}_1|^2 - |(\mathbf{X}_0 - \mathbf{X}_1)\cdot(\mathbf{X}_3 - \mathbf{X}_1)|^2}}{|\mathbf{X}_3 - \mathbf{X}_1|},$$
$$d_3 = \frac{\sqrt{|\mathbf{X}_0 - \mathbf{X}_2|^2|\mathbf{X}_1 - \mathbf{X}_2|^2 - |(\mathbf{X}_0 - \mathbf{X}_2)\cdot(\mathbf{X}_1 - \mathbf{X}_2)|^2}}{|\mathbf{X}_1 - \mathbf{X}_2|}.$$

With $\mathbf{X_0} = (3, \frac{5}{2})$, $\mathbf{X}_1 = (2, 0)$, $\mathbf{X}_2 = (2, 2)$, and $\mathbf{X}_3 = (4, 4)$, straightforward computations yield $d_1 = \dfrac{1}{2\sqrt{2}}$, $d_2 = \dfrac{1}{2\sqrt{5}}$, and $d_3 = 1$; hence, $\rho = \dfrac{1}{2\sqrt{5}}$.

5.1.13. If $|\overline{\mathbf{X}} - \mathbf{X}_0| = r$ then every neighborhood of $\overline{\mathbf{X}}$ contains points from $S_r(\mathbf{X}_0)$ and points from $S_r(\mathbf{X}_0)^c$; hence, $\overline{\mathbf{X}}$ is a boundary point of $S_r(\mathbf{X}_0)$. If $|\overline{\mathbf{X}} - \mathbf{X}_0| \neq r$, then $\overline{\mathbf{X}}$ is either in the interior or the exterior of $S_r(\mathbf{X}_0)$ so $\overline{\mathbf{X}}$ is not a boundary point of $S_r(\mathbf{X}_0)$. Therefore, $\partial S_r(\mathbf{X}_0) = \{\mathbf{X} \,\big|\, |\mathbf{X} - \mathbf{X}_0| = r\}$, so $\overline{S}_r(\mathbf{X}_0) = S_r(\mathbf{X}_0) \cup \partial S_r(\mathbf{X}_0) = \{\mathbf{X} \,\big|\, |\mathbf{X} - \mathbf{X}_0| \le r\}$.

5.1.14. Suppose that $\mathbf{X}_0 \in A$. Since A is open, $\mathbf{X}_0$ has a neighborhood $N \subset A$. Since $N \cap B = \emptyset$, $\mathbf{X}_0$ is not a limit point of B. Therefore, $A \cap \overline{B} = \emptyset$. Similarly, $\overline{A} \cap B = \emptyset$.

5.1.15. See the proof of Theorem 4.1.8.

5.1.16. By definition, $|\mathbf{X}_r - \overline{\mathbf{X}}| = \left[\sum_{i=1}^n (x_{ir} - \overline{x}_i)^2\right]^{1/2}$. Suppose that $\epsilon > 0$. If (A) $\lim_{r\to\infty} x_{ir} = \overline{x}_i$, $1 \le i \le n$, there is an integer R such that $|x_{ir} - \overline{x}_i| < \epsilon/\sqrt{n}$ for $1 \le i \le n$ if $r \ge R$. Then (B) $|\mathbf{X}_r - \overline{\mathbf{X}}| < \epsilon$ if $r \ge R$. Hence, (C) $\lim_{r\to\infty} \mathbf{X}_r = \overline{\mathbf{X}}$. Since $|x_{ir} - \overline{x}_i| \le |\mathbf{X}_r - \overline{\mathbf{X}}|$, (B) implies that $|x_{ir} - \overline{x}_i| < \epsilon$ if $r \ge R$; hence, (C) implies (A).

5.1.17. Suppose that $\{\mathbf{X}_r\}$ converges. Then the sequences $\{x_{ir}\}$, $1 \le i \le n$, all converge (Theorem 4.5.14). Therefore, if $\epsilon > 0$, there is an integer R such that $|x_{ir} - x_{is}| < \dfrac{\epsilon}{\sqrt{n}}$ for $1 \le i \le n = 1$ if $r, s \ge R$ (Theorem 4.1.13). This implies that (A) $|\mathbf{X}_r - \mathbf{X}_s| < \epsilon$ if r, $s \ge R$. Conversely, suppose that for every $\epsilon > 0$ there is an integer R satisfying (A). Then $|x_{ir} - x_{is}| < \epsilon$ for $1 \le i \le n = 1$ if r, $s \ge R$, so the sequences $\{x_{ir}\}$, $1 \le i \le n$, all converge (Theorem 4.1.13). Therefore, $\{\mathbf{X}_r\}$ converges (Theorem 4.5.14).

5.1.18. **(a)** $\lim_{r\to\infty} \mathbf{X}_r = \left(\lim_{r\to\infty} r \sin\dfrac{\pi}{r}, \lim_{r\to\infty} \cos\dfrac{\pi}{r}, \lim_{r\to\infty} e^{-r}\right) = (\pi, 1, 0)$.

(b) $\lim_{r\to\infty} \mathbf{X}_r = \left(\lim_{r\to\infty}\left(1 - \dfrac{1}{r^2}\right), \lim_{r\to\infty} \log\dfrac{r+1}{r+2}, \lim_{r\to\infty}\left(1 + \dfrac{1}{r}\right)^r\right) = (1, 0, e)$.

5.1.19. **(a)** $d(S)$ is the supremum of $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ where $-2 \le x_1, x_2 \le 2$, $-1 \le y_1, y_2 \le 1$, $0 \le z_1, z_2 \le 4$; to maximize this function let, for example, $x_1 = -2$, $x_2 = 2$, $y_1 = -1$, $y_2 = 1$, $z_1 = 0$, $z_2 = 4$; thus, $d(S) = \sqrt{4^2 + 2^2 + 4^2} = 6$.

(b) $d(S)$ is the length of the major axis; that is, $d(S) = 6$.

(c) $d(S)$ is the length of the longest side; that is $d(S) = |(4,4)-(2,0)| = |(2,4)| = 2\sqrt{5}$.

(d) $d(S)$ is the supremum of $\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$, where $-L \le x_i, y_i \le L$. To maximize this function let, for example, $x_i = -L$ and $y_i = L$, $1 \le i \le n$; thus, $d(S) = 2L\sqrt{n}$.

(e) Since S is unbounded, $d(S) = \infty$.

5.1.20. Since $S \subset \overline{S}$, (A) $d(S) \le d(\overline{S})$. If $\mathbf{X}$ and $\mathbf{Y}$ are in $\overline{S}$ and $\epsilon > 0$, there are points $\mathbf{X}_1, \mathbf{Y}_1$ in S such that $|\mathbf{X} - \mathbf{X}_1| < \epsilon$ and $|\mathbf{Y} - \mathbf{Y}_1| < \epsilon$. Then

$$|\mathbf{X} - \mathbf{Y}| \le |\mathbf{X} - \mathbf{X}_1| + |\mathbf{X}_1 - \mathbf{Y}_1| + |\mathbf{Y}_1 - \mathbf{Y}| < d(S) + 2\epsilon.$$

Hence, $d(\overline{S}) < d(S) + \epsilon$. Let $\epsilon \to 0+$ to conclude that $d(\overline{S}) \le d(S)$, which, with (A), implies that $d(S) = d(\overline{S})$.

5.1.21. Suppose that $\mathbf{X}_0$ is in S. If $S \ne \mathbb{R}^n$, there is an $\mathbf{X}_1 \notin S$. Let

$$H = \left\{\tau \,\middle|\, (1-t)\mathbf{X}_0 + t\mathbf{X}_1 \in S \text{ for } 0 \le t < \tau\right\}.$$

Since S is open it contains a neighborhood of $\mathbf{X}_0$, so $H \ne \emptyset$. Since $\mathbf{X}_1$ is not in S, $\tau \le 1$ for all τ in H. Let $\rho = \sup\ H$ and $\overline{\mathbf{X}} = (1-\rho)\mathbf{X}_0 + \rho\mathbf{X}_1$; then $\overline{\mathbf{X}}$ is a limit point of S and so in S, because S is closed. Since S is open, it contains some $\epsilon-$neighborhood of $\overline{\mathbf{X}}$, so $\rho + \epsilon/2$ is in H. This contradicts the definition of ρ. Hence, $S = R^n$.

5.1.22. If S_M has finitely many members for some M, then some point from S_M is in $\cap_{m=1}^{\infty} S_m$. Hence, we assume that S_m has infinitely many members for every m. Choose $\mathbf{X}_i$ in S_i so that $\mathbf{X}_i \ne \mathbf{X}_j$ if $i \ne j$. Then $\{\mathbf{X}_i\}$ is a bounded infinite set and has a limit point, by the Bolzano - Weierstrass theorem. Since $S_i \supset S_{i+1}$, $\overline{\mathbf{X}}$ is a limit point of every S_i. Since S_i is closed, $\overline{\mathbf{X}}$ is in each S_i, and so in $\cap_{i=1}^{\infty} S_i$. The conclusion does not hold if $S_m = [m, \infty)$, which is closed, but no bounded.

5.1.23. S_n is compact. If S_n is nonempty for all n, then $\cap_{n=1}^{\infty} S_n \ne \emptyset$ (Exercise 5.1.22), which implies that $U_1, U_2, \dots,$ do not cover S, a contradiction.

5.1.24. **(a)** First, note that $C_m \ne \emptyset$. To see this, suppose that $C_m = \emptyset$. Then $|\mathbf{X} - \mathbf{X}_0| \ge \text{dist}(\mathbf{X}_0, S) + 1/m$ for all $\mathbf{X} \in S$, contrary to the definition of dist$(\mathbf{X}_0, S)$. Moreover, C_m is closed. To see this, suppose that $\mathbf{Z}$ is a limit point of C_m. Then $\mathbf{Z}$ is also a limit point of of S, and therefore in S. Hence, if $\epsilon > 0$, there is an $\mathbf{X}$ in C_m such that $|\mathbf{X} - \mathbf{Z}| < \epsilon$, so $|\mathbf{Z} - \mathbf{X}_0| \le |\mathbf{Z} - \mathbf{X}| + |\mathbf{X} - \mathbf{X}_0| < \epsilon + \text{dist}(\mathbf{X}_0) + 1/m$. Letting $\epsilon \to 0$ shows that $Z \in C_m$. Therefore, C_m is compact.

Since $C_{m+1} \subset C_m$, there is an $\overline{\mathbf{X}}$ in $\bigcap_{m=1}^{\infty} C_m$ (Exercise 5.1.22). Since dist$(\mathbf{X}_0, S) \le |\overline{\mathbf{X}} - \mathbf{X}_0| \le \text{dist}(\mathbf{X}_0, S) + 1/m$, $m \ge 1$, $|\overline{\mathbf{X}} - \mathbf{X}_0| = \text{dist}(\mathbf{X}_0, S)$.

(b) Follows immediately from **(a)**, since $\overline{\mathbf{X}} \ne \mathbf{X}_0$ if $\mathbf{X}_0 \notin S$, and therefore $|\overline{\mathbf{X}} - \mathbf{X}_0| > 0$.

(c) In R^2, let $S = \left\{(x,y) \,\middle|\, x^2 + y^2 < 1\right\}$ and $\mathbf{X}_0 = (1,0)$; then dist$(\mathbf{X}_0, S) = 0$, but $|\mathbf{X} - \mathbf{X}_0| > 0$ if $\mathbf{X} \in S$.

5.1.25. **(a)** Let $\rho = \text{dist}(S, T)$. For each positive integer m, let $D_m = \left\{\mathbf{Z} \,\middle|\, \text{dist}(\mathbf{Z}, T) \le \rho + 1/m\right\}$. Since $T \subset D_m$, D_m is nonempty. Moreover, D_m is closed; to see this, suppose that $\overline{\mathbf{Z}}$ is a limit point of D_m and $\epsilon > 0$. Then there is a $\mathbf{Z}_0 \in D_m$ such that $|\overline{\mathbf{Z}} - \mathbf{Z}_0| < \epsilon$ and a $\mathbf{Z}_1$ in T such that $|\mathbf{Z}_0 - \mathbf{Z}_1| \le \rho + 1/m + \epsilon$. Hence, $|\overline{\mathbf{Z}} - \mathbf{Z}_1| < \rho + 1/m + 2\epsilon$. Since this is true for every $\epsilon > 0$, $\text{dist}(\overline{\mathbf{Z}}, T) \le \rho + 1/m$; that is, $\overline{\mathbf{Z}} \in D_m$. The triangle inequality shows that D_m is bounded, because T is; thus D_m is compact.

Now define $C_m = \left\{\mathbf{Z} \,\middle|\, \mathbf{Z} \in S \text{ and } \text{dist}(\mathbf{Z}, T) \le \rho + 1/m\right\}$.

Since S is closed and $C_m = D_m \cap S$, C_m is compact. Since $C_{m+1} \subset C_m$ and $C_m \ne \emptyset$, Exercise 5.1.22 implies that there is an $\overline{\mathbf{X}}$ in $\cap_{m=1}^{\infty} C_m$. This $\overline{\mathbf{X}}$ is in S and $\text{dist}(\overline{\mathbf{X}}, T) = \rho$. From Exercise 5.1.24, there is a $\overline{\mathbf{Y}}$ in T such that $|\overline{\mathbf{X}} - \overline{\mathbf{Y}}| = \rho$.

(b) If $S \cap T = \emptyset$, then $\overline{\mathbf{X}} \ne \overline{\mathbf{Y}}$, so $|\overline{\mathbf{X}} - \overline{\mathbf{Y}}| > 0$.

(c) In $\mathbb{R}^2$, let $S = \left\{(x, y) \,\middle|\, x^2 + y^2 < 1\right\}$ and $T = \left\{(x, y) \,\middle|\, (x-3)^2 + y^2 = 1\right\}$. Then $\text{dist}(S, T) = 1$, but $|\mathbf{X} - \mathbf{Y}| > 1$ if $\mathbf{X} \in S$ and $\mathbf{Y} \in T$.

In $\mathbb{R}$, let $S = \{2, 4, 6, \dots\}$ and $T = \bigcup_{n=2}^{\infty} \left[2n + \frac{1}{n}, 2n + 2 - \frac{1}{n}\right]$. Then $\text{dist}(S, T) = 0$, but $|\mathbf{X} - \mathbf{Y}| > 0$ if $\mathbf{X} \in S$ and $\mathbf{Y} \in T$.

5.1.26. **(a)** For each $\mathbf{X}$ in S there is an n-ball $B_{\mathbf{X}}$ of radius $r_{\mathbf{X}}$ centered at $\mathbf{X}$ such that $B_{\mathbf{X}} \subset U$. Let $B'_{\mathbf{X}}$ be the open n-ball about $\mathbf{X}$ of radius $r_{\mathbf{X}}/2$. From the Heine - Borel theorem finitely many of these, say $N'_{\mathbf{X}_1}, \dots, B'_{\mathbf{X}_k}$, cover S. Let $r = \min\left\{r_{\mathbf{X}_i}/2 \,\middle|\, 1 \le i \le k\right\}$. Now suppose that $\text{dist}(\mathbf{X}, S) \le r$. Then there is a point $\overline{\mathbf{X}}$ in S such that $|\mathbf{X} - \overline{\mathbf{X}}| = \text{dist}(\mathbf{X}, S)$ (Exercise 5.1.24), and $\overline{\mathbf{X}} \in N'_{\mathbf{X}_i}$ for some i. Now $|\mathbf{X} - \mathbf{X}_i| \le |\mathbf{X} - \overline{\mathbf{X}}| + |\overline{\mathbf{X}} - \mathbf{X}_i| \le r + r_{\mathbf{X}_i}/2 \le r_{\mathbf{X}_i}$, so $\mathbf{X}$ is in $B_{\mathbf{X}_i} \subset U$.

(b) Suppose that $|\mathbf{X}| \le M$ for every $\mathbf{X}$ in S. If $\mathbf{Y} \in S_r$, choose $\mathbf{X}$ in S so that $|\mathbf{X} - \mathbf{Y}| = \text{dist}(\mathbf{Y}, S)$. Then $|\mathbf{Y}| \le |\mathbf{Y} - \mathbf{X}| + |\mathbf{X}| \le r + M$; hence, S_r is bounded. To see that S_r is closed, suppose that $\overline{\mathbf{Y}}$ is a limit point of S_r and $\epsilon > 0$. Then there is a $\mathbf{Y}$ in S_r s uch that $|\mathbf{Y} - \overline{\mathbf{Y}}| < \epsilon$, and an $\mathbf{X}$ in S such that $\text{dist}(\mathbf{Y}, S) = |\mathbf{X} - \mathbf{Y}|$. Then $|\overline{\mathbf{Y}} - \mathbf{X}| \le |\overline{\mathbf{Y}} - \mathbf{Y}| + |\mathbf{Y} - \mathbf{X}| < \epsilon + r$, and so $\text{dist}(\overline{\mathbf{Y}}, S) < \epsilon + r$. Letting $\epsilon \to 0$, we see that $\text{dist}(\overline{\mathbf{Y}}, S) \le r$; therefore, $\overline{\mathbf{Y}} \in S_r$, and S_r is closed.

5.1.27. If $\mathbf{U} = (\mathbf{X}, \mathbf{Y})$, then $|\mathbf{U}|^2 = |\mathbf{X}|^2 + |\mathbf{Y}|^2$; hence, D is bounded, since D_1 and D_2 are. If $\mathbf{U}_0 = (\mathbf{X}_0, \mathbf{Y}_0)$ is a limit point of D and $\epsilon > 0$, there are points $\mathbf{X}$ in D_1 and $\mathbf{Y}$ in D_2 such that $|\mathbf{U} - \mathbf{U}_0| < \epsilon$, where $\mathbf{U} = (\mathbf{X}, \mathbf{Y})$. Therefore, $|\mathbf{X} - \mathbf{X}_0| < \epsilon$ and $|\mathbf{Y} - \mathbf{Y}_0| < \epsilon$, which implies that $\mathbf{X}_0 \in \overline{D}_1 = D_1$, and $\mathbf{Y}_0 \in \overline{D}_2 = D_2$; hence, $\mathbf{U}_0 \in D$. Hence, $\overline{D} = D$.

5.1.28. Suppose that $\mathbf{X}_0 \in A$. Since S is open, there is a neighborhood N_1 of $\mathbf{X}_0$ such that $N_1 \subset A \cup B$. Since $\mathbf{X}_0 \notin \overline{B}$ (because $A \cap \overline{B} = \emptyset$), there is a neighborhood N_2 of $\mathbf{X}_0$ such that $N_2 \cap B = \emptyset$. Now $N_1 \cap N_2 \subset A$. Hence, A is open. Similarly, B is open.

5.1.30. Suppose that $S = A \cup B$, where $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$. Since S^0 is connected, we can assume that $S^0 \subset A$. But then $S \subset \overline{S^0}$ (by definition of region) and $\overline{S^0} \subset \overline{A}$, so $S \subset \overline{A}$. Since $\overline{A} \cap B = \emptyset$, it follows that $S \cap B = \emptyset$; that is, $B = \emptyset$. Therefore, S is connected.

5.1.31. Suppose that S is a region in $\mathbb{R}$ and $a, b \in S$, with $a < b$. If $a < c < b$ and $c \notin S$, then $S = A \cup B$, where $A = (-\infty, c) \cap S$ and $B = (c, \infty) \cap S$ yields a "disconnection" of S. Hence, if a and b are in S, then (a, b) is also. This implies that S is an interval.

5.1.32. Since $\{\mathbf{X}_r\}$ is bounded, so are its sequences of components. Choose a subsequence of $\{\mathbf{X}_r\}$ for which the sequence of first components converges; this is possible, by Theorem 4.2.5**(a)**. Then pick a subsequence of this subsequence for which the sequence of second components converges; the associated sequence of first components still converges, by Theorem 4.2.2. Continuing in this way leads to a subsequence of $\{\mathbf{X}_r\}$ for which each component sequence converges. Now use Theorem 4.5.14.

5.2 CONTINUOUS REAL-VALUED FUNCTIONS OF n VARIABLES

5.2.1. **(a)** $\lim_{\mathbf{X}\to\mathbf{X}_0} = 3$; $|f(\mathbf{X})-10| = |3x+4y+z-12| = |3(x-1)+4(y-2)+(z-1)| \le \sqrt{26}|\mathbf{X}-\mathbf{X}_0|$, by Schwarz's inequality; hence, $|f(\mathbf{X})-10| < \epsilon$ if $|\mathbf{X}-\mathbf{X}_0| < \dfrac{\epsilon}{\sqrt{26}}$.

(b) Assume throughout that $(x,y) \in D_f = \left\{(x,y)\,\middle|\, x \ne y\right\}$. Then $f(x,y) = x^2+xy+y^2$, so $\lim_{\mathbf{X}\to\mathbf{X}_0} f(x) = 3$; $|f(\mathbf{X})-3| = |x^2+xy+y^2-3| = |(x-1)(x+2)+(y-1)(x+y+1)| \le \sqrt{(x+2)^2+(x+y+1)^2}|\mathbf{X}-\mathbf{X}_0|$, by Schwarz's inequality. If $|\mathbf{X}-\mathbf{X}_0| < 1$, then $\sqrt{(x+2)^2+(x+y+1)^2}| < 3\sqrt{2}\mathrm{e}$ and $|f(\mathbf{X})-3| < 3\sqrt{2}|\mathbf{X}-\mathbf{X}_0|$. Hence, if $|\mathbf{X}-\mathbf{X}_0| < \min\left(1, \dfrac{\epsilon}{3\sqrt{2}}\right)$, then $|f(\mathbf{X})-3| < \epsilon$.

(c) $\lim_{\mathbf{X}\to\mathbf{X}_0} f(\mathbf{X}) = 1$; if $\epsilon > 0$, there is a $\delta_1 > 0$ such that $\left|\dfrac{\sin u}{u} - 1\right| < \epsilon$ if $0 < |u| < \delta_1$. If $u = x+4y+2z = (x+2)+4(y-1)+2(z+1)$ then $|u| \le \sqrt{21}\,|\mathbf{X}-\mathbf{X}_0| < \delta_1$ if $|\mathbf{X}-\mathbf{X}_0| < \dfrac{\delta_1}{\sqrt{21}}$; hence, $|f(\mathbf{X})-1| < \epsilon$ if $|\mathbf{X}-\mathbf{X}_0| < \dfrac{\delta_1}{\sqrt{21}}$.

(d) $\lim_{\mathbf{X}\to\mathbf{X}_0} f(\mathbf{X}) = 0$; if $\epsilon > 0$, there is a $\delta > 0$ such that $|u^2 \log u| < \epsilon$ if $0 < |u| < \delta$. Since $f(\mathbf{X}) = |\mathbf{X}|^2 \log |X|$, $|f(\mathbf{X})| < \epsilon$ if $0 < |\mathbf{X}| < \delta$.

(e) Assume throughout that $(x,y) \in D_f = \left\{(x,y)\,\middle|\, x > y\right\}$. Then $|f(\mathbf{X})| = \sqrt{x-y}\left|\dfrac{\sin(x-y)}{x-y}\right| \le \sqrt{x-y} \le \sqrt{\sqrt{2}|\mathbf{X}|}$ by Schwarz's inequality, so $|f(\mathbf{X})| < \epsilon$ if $0 < |\mathbf{X}| < \dfrac{\epsilon^2}{\sqrt{2}}$.

(f) Since $\lim_{u\to\infty} ue^{-u} = 0$, for $\epsilon > 0$ there is an M such that $ue^{-u} < \epsilon$ if $u > M$. Therefore, $|f(\mathbf{X})| < \epsilon$ if $0 < |\mathbf{X}| < \dfrac{1}{M}$, so $\lim_{\mathbf{X}\to\mathbf{0}} f(\mathbf{X}) = 0$.

5.2.2. See the proof of Theorem 2.1.3.

5.2.3. **(a)** If $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon \quad \text{if} \quad 0 < \left[(x-x_0)^2 + (y-y_0)^2\right]^{1/2} < \delta.$$

There is a $\delta_1 > 0$ such that $|g(x) - y_0| < \delta/\sqrt{2}$ if $|x-x_0| < \delta_1$. Therefore, if $|x-x_0| < \min(\delta_1, \delta/\sqrt{2})$ then $\left[(x-x_0)^2 + (g(x)-y_0)^2\right]^{1/2} < \delta$, so $|f\,(x, g(x)) - L| < \epsilon$.

(b) Since $f(x,y)) = \dfrac{(y/x)}{1+(y/x)^2}$, $\lim_{x\to 0} f(x, y(x)) = \dfrac{a}{1+a^2}$ if $\lim_{x\to 0} \dfrac{y(x)}{x} = a$.

(c) Since $g(x,y(x)) = \dfrac{x(y/x)^4}{(1+x^4(y/x)^6)^3}$, $\lim_{x\to 0} g(x,y(x)) = 0$ if $\dfrac{y(x)}{x}$ remains bounded as $x\to 0$; however, if $y(x)=\sqrt{x}$, then $g(x,y(x)) = \dfrac{1}{x(1+x)^3}$, so $\lim_{x\to 0+} g(x,y(x)) = \infty$.

5.2.4.

(a) Henceforth, $\mathbf{X}\in D_f = \left\{(x,y,z)\,\middle|\, x+2y+4z\ne 0\right\}$; then $f(\mathbf{X}) = g(u(\mathbf{X}))$, where $g(u) = \dfrac{|\sin u|}{u^2} = \left|\dfrac{\sin u}{u}\right|\dfrac{1}{|u|}$ and $u(\mathbf{X}) = x+2y+4z$. Let M be an arbitrary real number. Since $\lim_{u\to 0} g(u) = 1\cdot\infty = \infty$, there is $\delta_1>0$ such that $g(u)>M$ if $0<|u|<\delta_1$. Since $\lim_{\mathbf{X}\to\mathbf{X}_0} u(\mathbf{X}) = 0$ there is a $\delta>0$ such that $0<|u(\mathbf{X})|<\delta_1$ if $|\mathbf{X}-\mathbf{X}_0|<\delta$; hence, $f(\mathbf{X})>M$ if $0<|\mathbf{X}-\mathbf{X}_0|<\delta$, so $\lim_{\mathbf{X}\to(2,-1,0)} f(\mathbf{X}) = \infty$.

(b) Henceforth, $\mathbf{X}\in D_f = \left\{(x,y)\,\middle|\, y<x\right\}$. Then $0<x-y<\sqrt{2}|\mathbf{X}|$ (Schwarz's inequality), so $\sqrt{x-y} < 2^{1/4}\sqrt{|\mathbf{X}|}$ and $f(\mathbf{X}) > \dfrac{1}{2^{1/4}\sqrt{|X|}}$. If $M>0$ and $|\mathbf{X}| < \dfrac{1}{\sqrt{2}M^2}$, then $f(\mathbf{X})>M$, so $\lim_{\mathbf{X}\to(0,0)} = \infty$.

(c) If $\mathbf{X}_n = \left(\frac{1}{n\pi},0\right)$, then $\lim_{n\to\infty} f(\mathbf{X}_n) = \lim_{n\to\infty}\sqrt{n\pi}\,\sin n\pi = 0$. If $\mathbf{X}_n = \left(\frac{1}{(2n+\frac{1}{2})\pi},0\right)$, then $\lim_{n\to\infty} f(\mathbf{X}_n) = \lim_{n\to\infty}\sqrt{(2n+\dfrac{1}{2})\pi}\,\sin\left(2n+\dfrac{1}{2}\right)\pi = \infty$. Hence, $\lim_{\mathbf{X}\to(0,0)} f(\mathbf{X})$ does not exist in the extended reals.

(d) Henceforth, $\mathbf{X}\in D_f = \left\{(x,y)\,\middle|\, x-2y\ne 0\right\}$. Then $f(\mathbf{X}) = -\dfrac{2y+x}{(x-2y)^2}$, so $\lim_{\mathbf{X}\to(2,1)} f(\mathbf{x}) = -\lim_{\mathbf{X}\to(2,1)}(2y+x)\lim_{\mathbf{X}\to(2,1)}\dfrac{1}{(x-2y)^2} = -4\cdot\infty = -\infty$.

(e) Henceforth, $\mathbf{X}\in D_f = \left\{(x,y,z)\,\middle|\, x+2y+4z\ne 0\right\}$; then $f(\mathbf{X}) = g(u(\mathbf{X}))$, where $g(u) = \dfrac{1}{u}\dfrac{\sin u}{u}$ and $u(\mathbf{X}) = x+2y+4z$. If $\{\mathbf{X}_n\}$ is sequence of points such that $u(\mathbf{X}_n)>0$ and $\mathbf{X}_n\to(2,-1,0)$ then $\lim_{n\to\infty} f(u(\mathbf{X}_n)) = \lim_{n\to\infty}\dfrac{1}{u(\mathbf{X}_n)}\dfrac{\sin u(\mathbf{X_n})}{u(\mathbf{X}_n)} = \infty\cdot 1 = \infty$. If $\{\mathbf{X}_n\}$ is sequence of points such that $u(\mathbf{X}_n)<0$ and $\mathbf{X}_n\to(2,-1,0)$ then $\lim_{n\to\infty} f(u(\mathbf{X}_n)) = \lim_{n\to\infty}\dfrac{1}{u(\mathbf{X}_n)}\dfrac{\sin u(\mathbf{X_n})}{u(\mathbf{X}_n)} = -\infty\cdot 1 = -\infty$. Hence, $\lim_{\mathbf{X}\to(2,-1,0)} f(\mathbf{X})$ does not exist in the extended reals.

5.2.5. **(a)** Since $x^2+2y^2+4z^2 \le 4|\mathbf{X}|^2$, $|f(\mathbf{X})| \le \dfrac{\log 4 + 2\log|\mathbf{X}|}{|\mathbf{X}|^2}$; $\lim_{|\mathbf{X}|\to\infty} f(\mathbf{X}) = 0$.

(b) $|f(\mathbf{X}).tex| \le \dfrac{1}{|\mathbf{X}|}$; $\lim_{|\mathbf{X}|\to\infty} f(\mathbf{X}) = 0$.

(c) $\lim_{x\to\infty} f(x,x) = \lim_{x\to\infty} e^{-4x^2} = 0$, while $\lim_{x\to\infty} f(x,-x) = 1$; therefore, $\lim_{|\mathbf{X}|\to\infty} f(\mathbf{X})$

does not exist.

(d) $f(\mathbf{X}) = e^{-|\mathbf{X}|^2}$, so $\lim_{|\mathbf{X}|\to\infty} f(\mathbf{X}) = 0$.

(e) $\lim_{x\to\infty} f(x,x) = 1$ while $\lim_{x\to\infty} f(x,0) = 0$; therefore, $\lim_{|\mathbf{X}|\to\infty} f(\mathbf{X})$ does not exist.

(a) $\lim_{|\mathbf{x}|\to\infty} f(\mathbf{X}) = \infty$ if D_f is unbounded and for every real number M, there is a number R such that $f(\mathbf{X}) > M$ if $|\mathbf{X}| > R$ and $\mathbf{X} \in D_f$.

(b) $\lim_{|\mathbf{x}|\to\infty} f(\mathbf{X}) = -\infty$ if D_f is unbounded and for every real number M, there is a number R such that $f(\mathbf{X}) < M$ if $|\mathbf{X}| > R$ and $\mathbf{X} \in D_f$.

5.2.7. Since $|f(\mathbf{X})| \le |X|^{a_1+\cdots+a_n-b}$, $\lim_{\mathbf{X}\to\mathbf{0}} f(\mathbf{X}) = 0$ if $a_1 + a_2 + \cdots + a_n > b$. If $c_1, c_2, \dots, c_n$ are constants, then

$$f(c_1r, c_2r, \dots, c_nr) = \frac{|c_1|^{a_1}|c_2|^{a_2}\cdots|c_n|^{a_n}}{(c_1^2 + c_2^2 + \cdots + c_n^2)^{b/2}} r^{a_1+a_2+\cdots+a_n-b}.$$

Therefore, f has zeros in every neighborhood of 0 if any $a_i > 0$ (set $c_i = 0$); however, by taking $c_j \ne 0$ $1 \le j \le n$, it can also be seen that f assumes arbitrarily large values in every neighborhood of 0 if (A) $a_1 + a_2 + \cdots + a_n < b$. Hence, $\lim_{X\to\mathbf{0}} f(\mathbf{X})$ does not exist if (A) holds with a least one a_i nonzero. If $a_1 = \cdots = a_n = 0$ and $b > 0$ then $\lim_{\mathbf{X}\to\mathbf{0}} f(\mathbf{X}) = \infty$. If $a_1 = \cdots = a_n = b = 0$, f is constant.

5.2.8. $g(x,0) = x^6 \to \infty$ as $|x| \to \infty$. If $a \ne 0$ then $g(x,ax) = \dfrac{x^{12}(a^4 + 1/x^2)^3}{x^{10}(a^4 + 1/x^10)} \to \infty$ as $|x| \to \infty$. However, $g(x, \sqrt{x}) = \dfrac{8x^6}{1 + x^8} \to 0$ as $|x| \to \infty$. Hence, $\lim_{|\mathbf{X}|\to\infty} g(\mathbf{X})$ does not exist.

5.2.11. Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \ne (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$; then $f(x,0) = x$ is continuous for all x and $f(0,y) = y$ is continuous for all y, but f is not continuous at $(0,0)$, since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (Example 5.2.3).

5.3 PARTIAL DERIVATIVES AND THE DIFFERENTIAL

5.3.1. **(a)** $h(t) = f(x+\phi_1 t, y+\phi_2 t) = (x+\phi_1 t)^2 + 2(x+\phi_1 t)(y+\phi_2 t)\cos(x+\phi_1 t)$;

$$\begin{aligned} h'(t) = {} & 2\phi_1(x+\phi_1 t) + 2\phi_1(y+\phi_2 t)\cos(x+\phi_1 t) + 2\phi_2(x+\phi_1 t)\cos(x+\phi_1 t) \\ & -2\phi_1(x+\phi_1 t)(y+\phi_2 t)\sin(x+\phi_1 t); \end{aligned}$$

$\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = (2x + 2y\cos x - 2xy\sin x)\phi_1 + 2\phi_2 x\cos x$; if $\acute{\mathbf{L}} = \left(\dfrac{1}{\sqrt{3}}, -\sqrt{\dfrac{2}{3}}\right)$, then $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = \dfrac{2}{\sqrt{3}}(x + y\cos x - xy\sin x) - 2\sqrt{\dfrac{2}{3}}(x\cos x)$.

(b) $h(t) = f(x+\phi_1 t, y+\phi_2 t, z+\phi_3 t) = \exp(-(x+\phi_1 t) + (y+\phi_2 t)^2 + 2(z+\phi_3 t))$;

$h'(t) = h(t)(-\phi_1 + 2\phi_2(y+\phi_2 t) + 2\phi_3)$; $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = e^{-x+y^2+2z}(-\phi_1 + 2y\phi_2 + 2\phi_3)$; if $\acute{\mathbf{L}} = \left(\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ then $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = \dfrac{(1-2y)}{\sqrt{3}} e^{-x+y^2+2z}$.

(c) $h(t) = \sum_{i=1}^{n} (x_i + \phi_i t)^2$; $h'(t) = 2\sum_{i=1}^{n} \phi_i x_i$; $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = 2\sum_{i=1}^{n} \phi_i x_i$; if $\acute{\mathbf{L}} = \left(\dfrac{1}{\sqrt{n}}, \dfrac{1}{\sqrt{n}}, \cdots, \dfrac{1}{\sqrt{n}}\right)$, then $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = \dfrac{2}{\sqrt{n}}(x_1 + x_2 + \cdots + x_n)$.

(d) $h(t) = f(x+\phi_1 t, y+\phi_2 t, z+\phi_3 t) = \log(1+x+y+z+(\phi_1+\phi_2+\phi_3)t)$; $h'(t) = \dfrac{\phi_1+\phi_2+\phi_3}{1+x+y+z+(\phi_1+\phi_2+\phi_3)t}$; $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = \dfrac{\phi_1+\phi_2+\phi_3}{1+x+y+z}$; if $\acute{\mathbf{L}} = (0, 1, 0)$ then $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = \dfrac{1}{1+x+y+z}$.

5.3.2. $h(t) = f(x+\phi_1 t, y+\phi_2 t) = \phi_1\phi_2 \sin\phi_1 t$; $h'(t) = \phi_1^2\phi_2\cos\phi_1 t$; $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = \phi_1^2\phi_2$.

5.3.3. **(a)** $h(t) = f(x+\phi_1 t, y+\phi_2 t, z+\phi_3 t) = \sin[\pi(x+\phi_1 t)(y+\phi_2 t)(z+\phi_3 t)]$; $h'(t) = \pi g(t)\cos[\pi(x+\phi_1 t)(y+\phi_2 t)(z+\phi_3 t)]$, with

$$g(t) = \phi_1(y+\phi_2 t)(z+\phi_3 t) + \phi_2(x+\phi_1 t)(z+\phi_3 t) + \phi_3(x+\phi_1 t)(y+\phi_2)t;$$

$\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = \pi g(0)\cos\pi xyz = \pi(\phi_1 yz + \phi_2 xz + \phi_3 xy)\cos\pi xyz$; if $\acute{\mathbf{L}} = \dfrac{\mathbf{X}_1 - \mathbf{X}_0}{|\mathbf{X}_1 - \mathbf{X}_0|} = \left(\dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$, then $\dfrac{\partial f(1,1,-2)}{\partial \acute{\mathbf{L}}} = -\dfrac{5\pi}{\sqrt{6}}$.

(b) $h(t) = f(x+\phi_1 t, y+\phi_2 t, z+\phi_3 t) = \exp -((x+\phi_1 t)^2 + (y+\phi_2 t)^2 + 2(z+\phi_3 t))$; $h'(t) = -2h(t)(\phi_1(x+\phi_1 t) + \phi_2(y+\phi_2 t) + \phi_3)$; $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = -2e^{-(x^2+y^2+2z)}(\phi_1 x + \phi_2 y + \phi_3)$; if $\acute{\mathbf{L}} = \dfrac{\mathbf{X}_1 - \mathbf{X}_0}{|\mathbf{X}_1 - \mathbf{X}_0|} = (1, 0, 0)$ then $\dfrac{\partial f(1,0,-1)}{\partial \acute{\mathbf{L}}} = 2e$.

(c) $h(t) = f(x+\phi_1 t, y+\phi_2 t, z+\phi_3 t) = \log(1+x+y+z+(\phi_1+\phi_2+\phi_3)t)$; $h'(t) = \dfrac{\phi_1+\phi_2+\phi_3}{1+x+y+z+(\phi_1+\phi_2+\phi_3)t}$; $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = \dfrac{\phi_1+\phi_2+\phi_3}{1+x+y+z}$; if $\acute{\mathbf{L}} = \dfrac{\mathbf{X}_1 - \mathbf{X}_0}{|\mathbf{X}_1 - \mathbf{X}_0|} = \left(\dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}}\right)$ then $\dfrac{\partial f(1,0,1)}{\partial \acute{\mathbf{L}}} = 0$.

(d) $h(t) = \left(\sum_{i=1}^{n}(x_i + t\phi_i)^2\right)^2$; $h'(t) = 4\left(\sum_{i=1}^{n}(x_i + t\phi_i)^2\right)\sum_{i=1}^{n}\phi_i(x_i + t\phi_i)$; $\dfrac{\partial f(\mathbf{X})}{\partial \acute{\mathbf{L}}} = h'(0) = \sum_{i=1}^{n} \acute{\mathbf{L}} \cdot \mathbf{X}$; $\dfrac{\partial f(\mathbf{0})}{\partial \acute{\mathbf{L}}} = 0$.

5.3.4. If $z_0 = f(x_0, y_0)$, then $z(t) = f(x_0 + t\phi_1, y_0 + t\phi_2)$ represents a curve through (x_0, y_0, z_0) in the plane determined by the unit vectors $\acute{\mathbf{L}}$ and $\mathbf{k}$; $\dfrac{\partial f(x_0, y_0)}{\partial \acute{\mathbf{L}}}$ is the slope of the curve at (x_0, y_0, z_0).

(a) $f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$ and $f_y(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$. If $(x,y)\ne(0,0)$ then $f_x(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$ and $f_y(x,y) = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$. $f_{xx}(0,0) = \lim_{x\to 0}\frac{f_x(x,0)-f_x(0,0)}{x} \lim_{x\to 0}\frac{0-0}{x} = 0$; $f_{yy}(0,0) = \lim_{y\to 0}\frac{f_y(0,y)-f_y(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$; $f_{xy}(0,0) = \lim_{y\to 0}\frac{f_x(0,y)-f_x(0,0)}{y} = \lim_{y\to 0}\frac{-y-0}{y} = -1$; $f_{yx}(0,0) = \lim_{x\to 0}\frac{f_y(x,0)-f_y(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$.

(b) $f_x(x,0) = \lim_{x\to 0}\frac{f(x+h,0)-f(x,0)}{h} = \lim_{x\to 0}\frac{0-0}{x} = 0$ for all x and $f_y(0,y) = \lim_{y\to 0}\frac{f(0,y+k)-f(0,y)}{k} = \lim_{y\to 0}\frac{0-0}{y} = 0$ for all y. If $y\ne 0$ then $f_x(0,y) = \lim_{x\to 0}\frac{f(x,y)-f(0,y)}{x} = \lim_{x\to 0}\left(x\tan^{-1}\frac{y}{x} - \frac{y^2}{x}\tan^{-1}\frac{x}{y}\right) = -\lim_{x\to 0}\frac{y^2}{x}\tan^{-1}\frac{x}{y} = -y$, by L'Hospital's rule and the boundedness of $\tan^{-1}u$. If $x\ne 0$ then $f_y(x,0) = \lim_{y\to 0}\frac{f(x,y)-f(x,0)}{y} = \lim_{y\to 0}\left(\frac{x^2}{y}\tan^{-1}\frac{y}{x} - y\tan^{-1}\frac{x}{y}\right) = \lim_{y\to 0}\frac{x^2}{y}\tan^{-1}\frac{y}{x} = x$, by L'Hospital's rule and the boundedness of $\tan^{-1}u$. Now $f_{xx}(0,0) = \lim_{x\to 0}\frac{f_x(x,0)-f_x(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$; $f_{yy}(0,0) = \lim_{y\to 0}\frac{f_y(0,y)-f_y(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$; $f_{xy}(0,0) = \lim_{y\to 0}\frac{f_x(0,y)-f_x(0,0)}{y} = \lim_{y\to 0}\frac{-y-0}{y} = -1$; $f_{yx}(0,0) = \lim_{x\to 0}\frac{f_y(x,0)-f_y(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$.

5.3.9. Assume throughout that $(x,y)\in S$. Differentiating $u_x = v_y$ with respect to x yields $u_{xx} = v_{yx}$. Differentiating $u_y = -v_x$ with respect to y yields $u_{yy} = -v_{xy}$. Therefore, $u_{xx}+u_{yy} = v_{yx}-v_{xy} = 0$ (Theorem 5.3.3).

Differentiating $u_x = v_y$ with respect to y yields $u_{xy} = v_{yy}$. Differentiating $u_y = -v_x$ with respect to x yields $-u_{yx} = v_{xx}$. Therefore, $v_{xx}+v_{yy} = u_{xy}-u_{yx} = 0$ (Theorem 5.3.3).

5.3.10. Apply Theorem 5.3.3 to f as a function of x_i and x_j, holding the other variables fixed.

5.3.11. $\mathbf{X}\in S$ throughout this proof. First consider $r=2$. The conclusion is obvious if $x_{i_1} = x_{i_2}$, and it follows from Exercise 5.3.10 if $x_{i_1}\ne x_{i_2}$. Now suppose that $r>2$ and the proposition is true with r replaced by $r-1$. If $x_{i_r} = x_{j_r}$, then $\{x_{i_1},\dots,x_{i_{r-1}}\}$ is a rearrangement of $\{x_{j_1},\dots,x_{j_{r-1}}\}$, so $f_{x_{i_1},\dots,x_{i_{r-1}}}(\mathbf{X}) = f_{x_{j_1},\dots,x_{j_{r-1}}}(\mathbf{X})$ by the induction assumption, and differentiating both sides of this equation with respect to $x_{i_r} = x_{j_r}$ yields $f_{x_{i_1},x_{i_2}\dots,x_{i_r}}(\mathbf{X}) = f_{x_{j_1},x_{j_2}\dots,x_{j_r}}(\mathbf{X})$. If $x_{i_r}\ne x_{j_r}$ then $x_{j_r} = x_{i_k}$ for some k in $\{1,\dots,r-1\}$. Let $\{x_{p_1},x_{p_2},\dots,x_{p_r}\}$ be the rearrangement of $\{x_{i_1},x_{i_2},\dots,x_{i_r}\}$ obtained by interchanging i_r and i_k. Then (A) $f_{x_{i_1},x_{i_2}\dots,x_{i_r}}(\mathbf{X}) = f_{x_{p_1},x_{p_2}\dots,x_{p_r}}(\mathbf{X})$, by

Exercise 5.3.10. However, now $x_{p_r} = x_{j_r}$, so $f_{x_{p_1},x_{p_2}\dots,x_{p_r}}(\mathbf{X}) = f_{x_{j_1},x_{j_2}\dots,x_{j_r}}(\mathbf{X})$ by the previous argument. This and (A) complete the induction.

5.3.12. Suppose that $n \ge 2$ and $P_0, P_1, \dots, P_n$ are true. Then

$$\begin{aligned}(z_1 + z_2 + \cdots + z_{n+1})^r &= [(z_1 + z_2 + \cdots + z_n) + z_{n+1}]^r \\ &= \sum_{j=0}^{r} \binom{r}{j} (z_1 + z_2 + \cdots + z_n)^j z_{n+1}^{r-j} \quad (\text{by } P_2) \\ &= \sum_{j=0}^{r} \binom{r}{j} \left[\sum_{j} \frac{j!}{j_1! j_2! \dots j_n!} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} \right] z_{n+1}^{r-j}\end{aligned}$$

by $P_0, P_1, \dots, P_n$. Since $\binom{r}{j} \dfrac{r!}{j!(r-j)!}$, this can be rewritten as

$$\begin{aligned}(z_1 + z_2 + \cdots + z_{n+1})^r &= r! \sum_{j=0}^{r} \sum_{j} \left(\frac{z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}}{j_1! j_2! \cdots j_n!} \right) \frac{z_{n+1}^{r-j}}{(r-j)!} \\ &= r! \sum_{r} \frac{z_1^{r_1} z_2^{r_2} \cdots z_{n+1}^{r_{n+1}}}{r_1! r_2! \cdots r_{n+1}!},\end{aligned}$$

which is P_{n+1}.

(b) There is a one-to-one correspondence between these n tuples and the products in the expansion of $(z_1 + z_2 + \cdots + z_n)^r$ that contain r_1 factors equal to x_1, r_2 factors equal to $x_2, \dots, r_n$ factors equal to x_n. The number of such products is $\dfrac{r!}{r_1! r_2! \cdots r_n!}$, from **(a)**.

(c) Follows from **(b)**.

5.3.13. Let $E(\mathbf{X}) = \begin{cases} \dfrac{f(\mathbf{X}) - f(\mathbf{X}_0) - d_{\mathbf{X}_0}(\mathbf{X} - \mathbf{X}_0)}{|\mathbf{X} - \mathbf{X}_0|}, & \mathbf{X} \ne \mathbf{X}_0, \\ 0, & \mathbf{X} = \mathbf{X}_0, \end{cases}$ and apply Theorem 5.3.7 and the definition of the differential.

5.3.14. If $\Phi = (\phi_1, \phi_2)$, then $\dfrac{\partial f(0,0)}{\partial \Phi} = h'(0)$, where $h(t) = \begin{cases} \dfrac{t\phi_1^2\phi_2}{t^4\phi_1^6 + 2\phi_2^2}, & t \ne 0, \\ 0, & t = 0. \end{cases}$

If $\Phi = (1,0)$ then $h \equiv 0$ and $h'(0) = 0$. If $\phi_2 \ne 0$, then $h'(0) = \lim_{t\to 0} \dfrac{\phi_1^2\phi_2}{t^4\phi_1^6 + 2\phi_2^2} = \dfrac{\phi_1^2}{2\phi_2}$.

Therefore, $\dfrac{\partial f(0,0)}{\partial \Phi}$ exists for every Φ. However, f is not continuous at $(0,0)$, since $\lim_{y\to 0+} f(\sqrt{y}, y)\dfrac{1}{2} \ne f(0,0)$.

5.3.15. Choose $r > 0$ so that $S_r(x_0, y_0)$ is in the neighborhood and $[f_x(x,y)| \le M_1$ and $|f_y(x,y)| \le M_2$ if $(x,y) \in S_r(x_0,y_0)$. If $(x,y) \in S_r(x_0,y_0)$, then

$$\begin{aligned}f(x,y) - f(x_0,y_0) &= [f(x,y) - f(x_0,y)] + [f(x_0,y) - f(x_0,y_0)] \\ &= f_x(\widehat{x}, y)(x - x_0) + f_y(x_0, \widehat{y})(y - y_0),\end{aligned}$$

where $\widehat{x}$ is between x and x_0 and $\widehat{y}$ is between y and y_0. (By the mean value theorem, applied first with respect to x, with y fixed, and then with respect to y, with x fixed.) Therefore, by Schwarz's inequality,

$$|f(x,y)-f(x_0,y_0)| \le \sqrt{M_1^2+M_2^2}\,\sqrt{(x-x_0)^2+(y-y_0)^2},$$

which implies the result.

5.3.16. **(a)**

$$\begin{aligned} f(x,y)-f(1,2) &= 2(x^2-1)+3(xy-2)+(y^2-4)\\ &= 2(x+1)(x-1)+3((x-1)y+(y-2))+(y+2)(y-2)\\ &= (2x+3y+2)(x-1)+(y+5)(y-2)\\ &= 10(x-1)+7(y-2)+(2x+3y-10)(x-1)+(y-2)(y-2); \end{aligned}$$

therefore, if $(x,y)\ne(1,2)$,

$$\frac{|f(x,y)-f(1,2)-10(x-1)-7(y-2)|}{\sqrt{(x-1)^2+(y-2)^2}} \le \sqrt{(2x+3y-10)^2+(y-2)^2},$$

by Schwarz's inequality. Since $\lim_{(x,y)\to(1,2)} \sqrt{(2x+3y-10)^2+(y-2)^2}=0$, f is differentiable at $(1,2)$.

(b)

$$\begin{aligned} f(x,y,z)-f(1,1,1) &= 2(x^2-1)+3(x-1)+4(yz-1)\\ &= (2x+5)(x-1)+4(y-1)+4y(z-1)\\ &= 7(x-1)+4(y-1)+4(z-1)+2(x-1)^2+4(z-1)(y-1); \end{aligned}$$

therefore, if $(x,y,z)\ne(1,1,1)$,

$$\frac{|f(x,y,z)-f(1,1,1)-7(x-1)-4(y-1)-4(z-1)|}{\sqrt{(x-1)^2+(y-1)^2+(z-1)^2}} \le 2\sqrt{(x-1)^2+4(z-1)^2},$$

by Schwarz's inequality. Since $\lim_{(x,y)\to(1,1,1)} \sqrt{(x-1)^2+4(z-1)^2}=0$, f is differentiable at $(1,1,1)$.

(c)

$$\begin{aligned} f(\mathbf{X})-f(\mathbf{X}_0) &= \sum_{i=1}^n (x_i^2-x_{i0}^2) = \sum_{i=1}^n (x_i+x_{i0})(x_i-x_{i0})\\ &= 2\sum_{i=1}^n x_{i0}(x_i-x_{i0}) + \sum_{i=1}^n (x_i-x_{i0})^2; \end{aligned}$$

therefore,

$$\frac{f(\mathbf{X})-f(\mathbf{X}_0)-2\displaystyle\sum_{i=1}^n x_{i0}(x_i-x_{i0})}{|\mathbf{X}-\mathbf{X}_0|} = |\mathbf{X}-\mathbf{X}_0|,$$

so f is differentiable at $\mathbf{X}_0$

5.3.17. Write

$$f(x,y)-f(x_0,y_0)=f(x,y)-f(x_0,y)+\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}(y-y_0). \qquad \text{(A)}$$

Holding y fixed and applying the mean value theorem with respect to x yields $f(x,y)-f(x_0,y)=f_x(\widehat{x}(y),y)$, where $\widehat{x}(y)$ is between x and x_0. Therefore,

$$\begin{aligned} f(x,y)-f(x_0,y) &= [f_x(x_0,y_0)+(f_x(\widehat{x},y)-f_x(x_0,y_0))](x-x_0) \\ &= [f_x(x_0,y_0)+\epsilon_1(x,y)](x-x_0), \end{aligned} \qquad \text{(B)}$$

where $\lim_{(x,y)\to(x_0,y_0)} \epsilon_1(x,y)=0$ because f_x is continuous at (x_0,y_0) From the definition of $f_y(x_0,y_0)$,

$$\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}=f_y(x_0,y_0)+\epsilon_2(y), \qquad \text{(C)}$$

where $\lim_{y\to y_0}\epsilon_2(y)=0$. Now (A), (B), (C), and Schwarz's inequality imply that if $(x,y)\neq(x_0,y_0)$ then

$$\frac{|f(x,y)-f(x_0,y_0)-f_x(x_0,y_0)(x-x_0)-f_y(x_0,y_0)(y-y_0)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\le E(x,y),$$

where $E(x,y)=\sqrt{\epsilon_1^2(x,y)+\epsilon_2^2(y)}$. Since $\lim_{(x,y)\to(x_0,y_0)}E(x,y)=0$, f is differentiable at (x_0,y_0).

5.3.19. **(a)** It is given that

$$\lim_{\mathbf{X}\to\mathbf{X}_0}\frac{f(\mathbf{X})-f(\mathbf{X}_0)-\sum_{i=1}^n f_{x_i}(\mathbf{X}_0)(x_i-x_{i0})}{|\mathbf{X}-\mathbf{X}_0|}=0.$$

Let $\mathbf{X}=\mathbf{X}_0+t\acute{\mathbf{L}}$; then

$$\lim_{t\to 0}\frac{f(\mathbf{X}_0+t\acute{\mathbf{L}})-f(\mathbf{X}_0)-\sum_{i=1}^n f_{x_i}(\mathbf{X}_0)t\phi_i}{t}=0;$$

that is,

$$\frac{\partial f(\mathbf{X}_0)}{\partial\acute{\mathbf{L}}}=\lim_{t\to 0}\frac{f(\mathbf{X}_0+t\acute{\mathbf{L}})-f(\mathbf{X}_0)}{t}=\sum_{i=1}^n f_{x_i}(\mathbf{X}_0)\phi_i.$$

(b) If $\sum_{i=1}^n f_{x_i}^2(X_0)=0$, then $\frac{\partial f(\mathbf{X}_0)}{\partial\acute{\mathbf{L}}}=0$ for every $\acute{\mathbf{L}}$. If $\sum_{i=1}^n f_{x_i}^2(X_0)\neq 0$, the maximum is attained with

$$\phi_i=f_{x_i}(X_0)\left(\sum_{j=1}^n f_{x_1}^2(X_0)\right)^{-1/2}, \quad 1\le i\le n.$$

To see this, use Lemma 5.1.5.

5.3.20. First note that $|g(u)| \le u^2$ for all u and

$$g'(u) = \begin{cases} 2u\sin\dfrac{1}{u} - \cos\dfrac{1}{u}, & u \ne 0 \\ \lim\limits_{u\to 0}\dfrac{g(u)}{u} = 0, & u = 0; \end{cases}$$

hence, $g'(u)$ exists for all u, but is discontinuous at $u = 0$. Now,

$$|f(\mathbf{X}) - f(\mathbf{0}) - 0\cdot x_1 - 0\cdot x_2 - \cdots - 0\cdot x_n| \le |g(x_1)| + |g(x_2)| + \cdots + |g(x_n)| \le |\mathbf{X}|^2,$$

so

$$\lim_{\mathbf{X}\to\mathbf{0}} \frac{|f(\mathbf{X}) - f(\mathbf{0}) - 0\cdot x_1 - 0\cdot x_2 - \cdots - 0\cdot x_n|}{|\mathbf{X}|} = 0;$$

that is, f is differentiable at $(0, 0, \dots, 0)$. Since $f_{x_i} = g'(x_i)$, f_{x_i} is discontinuous at $(0, 0, \dots, 0)$.

5.3.21. **(a)** $B(h) = \phi(x_0 + h) - \phi(x_0)$, where $\phi(x) = f(x, y_0 + k) - f(x, y_0)$. Since $\phi'(x) = f_x(x, y_0 + k) - f_x(x, y_0)$ the mean value theorem implies that (A) $B(h) = [f_x(\widehat{x}, y_0 + h) - f_x(\widehat{x}, y_0)]\,h$ where $\widehat{x}$ is between x_0 and $x_0 + h$. Since f_x is differentiable at (x_0, y_0),

$$\begin{aligned} f_x(\widehat{x}, y_0 + h) &= f_x(x_0, y_0) + f_{xx}(x_0, y_0)(\widehat{x} - x_0) + f_{xy}(x_0, y_0)h + a_1(h) \\ f_x(\widehat{x}, y_0) &= f_x(x_0, y_0) + f_{xx}(x_0, y_0)(\widehat{x} - x_0) + b_1(h), \end{aligned}$$

where $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{a_1(h)}{h} = \lim_{(x,y)\to(x_0,y_0)} \frac{b_1(h)}{h} = 0$. Therefore, from (A), (B) $B(h) = f_{xy}(x_0, y_0)h^2 + E_1(h)h$, with $E_1(h) = a_1(h) - b_1(h)$, so $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{E_1(h)}{h} = 0$.

(b) $B(h) = \psi(x_0 + h) - \psi(x_0)$, where $\psi(y) = f(x_0 + h, y) - f(x_0, y)$. Since $\psi'(y) = f_y(x_0 + h, y) - f_x(x_0, y)$ the mean value theorem implies that (C) $B(h) = \left[f_y(x_0 + h, \widehat{y}) - f_y(x_0, \widehat{y})\right] h$ where $\widehat{y}$ is between y_0 and $y_0 + h$. Since f_y is differentiable at (x_0, y_0),

$$\begin{aligned} f_y(x_0 + h, \widehat{y}) &= f_y(x_0, y_0) + f_{yx}(x_0, y_0)h + f_{yy}(x_0, y_0)(\widehat{y} - y_0) + a_2(h) \\ f_y(x_0, \widehat{y}) &= f_y(x_0, y_0) + f_{yy}(x_0, y_0)(\widehat{y} - y_0) + b_2(h), \end{aligned}$$

where $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{a_2(h)}{h} = \lim_{(x,y)\to(x_0,y_0)} \frac{b_2(h)}{h} = 0$. Therefore, from (C), (D) $B(h) = f_{yx}(x_0, y_0)h^2 + E_2(h)h$, with $E_2(h) = a_2(h) - b_2(h)$, so $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{E_2(h)}{h} = 0$.

(c) From (B) and (D) $f_{xy}(x_0, y_0)h^2 + E_1(h)h = f_{yx}(x_0, y_0)h^2 + E_2(h)h$; $f_{xy}(x_0, y_0) - f_{yx}(x_0, y_0) = \dfrac{E_2(h) - E_1(h)}{h}$; since the right side $\to 0$ as $h \to 0$ and the left side is independent of h, $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

5.3.22. **(a)** Apply the result of Exercise 5.3.21 to f as a function of x_i and x_j, holding the other variables fixed.

(b) $\mathbf{X} \in S$ throughout this proof. We want to show that $f_{x_{i_1}x_{i_2},\dots,x_{i_r}}(\mathbf{X}) = f_{x_{j_1}x_{j_2},\dots,x_{j_r}}(\mathbf{X})$ if each of the variables $x_1, x_2, \dots, x_n$ appears the same number of times in $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$ and $\{x_{j_1}, x_{j_2}, \dots, x_{j_r}\}$. First consider $r = 2$. The conclusion is obvious if $x_{i_1} = x_{i_2}$, and it follows from **(a)** if $x_{i_1} \ne x_{i_2}$. Now suppose that $r > 2$ and the proposition is true with r replaced by $r-1$. If $x_{i_r} = x_{j_r}$, then $\{x_{i_1}, \dots, x_{i_{r-1}}\}$ is a rearrangement of $\{x_{j_1}, \dots, x_{j_{r-1}}\}$, so $f_{x_{i_1},\dots,x_{i_{r-1}}}(\mathbf{X}) = f_{x_{j_1},\dots,x_{j_{r-1}}}(\mathbf{X})$ by the induction assumption, and differentiating both sides of this equation with respect to $x_{i_r} = x_{j_r}$ yields $f_{x_{i_1},x_{i_2}\dots,x_{i_r}}(\mathbf{X}) = f_{x_{j_1},x_{j_2}\dots,x_{j_r}}(\mathbf{X})$. If $x_{i_r} \ne x_{j_r}$ then $x_{j_r} = x_{i_k}$ for some k in $\{1, \dots, r-1\}$. Let $\{x_{p_1}, x_{p_2}, \dots, x_{p_r}\}$ be the rearrangement of $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$ obtained by interchanging i_r and i_k. Then (A) $f_{x_{i_1},x_{i_2}\dots,x_{i_r}}(\mathbf{X}) = f_{x_{p_1},x_{p_2}\dots,x_{p_r}}(\mathbf{X})$, by **(a)**. However, now $x_{p_r} = x_{j_r}$, so $f_{x_{p_1},x_{p_2}\dots,x_{p_r}}(\mathbf{X}) = f_{x_{j_1},x_{j_2}\dots,x_{j_r}}(\mathbf{X})$ by the previous argument. This and (A) complete the induction.

5.3.23. The three points lie on a line if and only if there are constants A and B such that $y_i - Ax_i - B = 0$ $(i = 1, 2, 3)$. This is equivalent to the condition that

$$0 = \begin{vmatrix} y_0 & x_0 & 1 \\ y_1 & x_1 & 1 \\ y_2 & x_2 & 1 \end{vmatrix} = \begin{vmatrix} y_0 & x_0 & 1 \\ y_1 - y_0 & x_1 - x_0 & 0 \\ y_2 - y_0 & x_2 - x_0 & 0 \end{vmatrix},$$

which is equivalent to the stated condition.

5.3.25. Since also

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0,$$

it follows that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{(f(x_0,y_0) - a) + (f_x(x_0,y_0) - b)(x-x_0) + (f_y(x_0,y_0) - c)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Therefore, the problem reduces to showing that if α, β, and γ are constants such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\alpha + \beta(x-x_0) + \gamma(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0, \qquad \text{(A)}$$

then $\alpha = \beta = \gamma = 0$. Since (A) implies that $\alpha + \lim_{(x,y)\to(x_0,y_0)} (\beta(x-x_0) + \gamma(y-y_0)) = 0$, it follows that $\alpha = 0$, so (A) reduces to $\lim_{(x,y)\to(x_0,y_0)} \dfrac{\beta(x-x_0) + \gamma(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$. Therefore, $\lim_{x\to x_0} \beta \dfrac{x-x_0}{|x-x_0|} = 0$, so $\beta = 0$, and $\lim_{y\to y_0} \gamma \dfrac{y-y_0}{|y-y_0|} = 0$, so $\gamma = 0$.

5.4 THE CHAIN RULE AND TAYLOR'S THEOREM

5.4.1. $\mathbf{X}_0 \in D_f^0$ and $\mathbf{U}_0 \in D_{\mathbf{G}}^0$, by definition of differentiability. Therefore, $B_\rho(\mathbf{X}_0) \subset D_f$ and $B_r(\mathbf{U}_0) \subset D_g$ for some $\rho, r > 0$. Since $\mathbf{G}$ is differentiable at $\mathbf{U}_0$, $\mathbf{G}$ is continuous at $\mathbf{U}_0$, so there is a δ such that $0 < \delta < r$ and $|\mathbf{G}(\mathbf{U}) - \mathbf{G}(\mathbf{U}_0)| = |\mathbf{G}(\mathbf{U}) - \mathbf{X}_0| < \rho$ if $|\mathbf{U} - \mathbf{U}_0| < \delta$. This means that $f(\mathbf{G}(\mathbf{U}))$ is defined if $|\mathbf{U} - \mathbf{U}_0| < \delta$; that is, $\mathbf{U}_0 \in D_h^0$.

5.4.2. **(a)** Let $\mathbf{U} = (u, v)$ and $\mathbf{X} = (x, y)$.

First method:

$\mathbf{U}_0 = (0, 1)$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial u} = ve^{u+v-1}$;

$\dfrac{\partial g_1(\mathbf{U}_0)}{\partial u} = 1$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial v} = e^{u+v-1} + ve^{u+v-1}$;

$\dfrac{\partial g_1(\mathbf{U}_0)}{\partial v} = 2$;

$d_{\mathbf{U}_0} g_1 = du + 2\,dv$;

$\dfrac{\partial g_2(\mathbf{U})}{\partial u} = -e^{-u+v-1}$;

$\dfrac{\partial g_2(\mathbf{U}_0)}{\partial u} = -1$;

$\dfrac{\partial g_2(\mathbf{U})}{\partial v} = e^{-u+v-1}$;

$\dfrac{\partial g_2(\mathbf{U}_0)}{\partial v} = 1$;

$d_{\mathbf{U}_0} g_2 = -\,du + \,dv$;

$\mathbf{X}_0 = (g_1(\mathbf{U}_0), g_2(\mathbf{U}_0)) = (1, 1)$;

$f_x(\mathbf{X}) = 6x + 4y^2 + 3$; $f_x(\mathbf{X}_0) = 13$;

$f_y(\mathbf{X}) = 8xy$; $f_y(\mathbf{X}_0) = 8$;

$d_{\mathbf{U}_0} h = f_x(\mathbf{X}_0)\, d_{\mathbf{U}_0} g_1 + f_y(\mathbf{X}_0)\, d_{\mathbf{U}_0} g_2 = 13(\,du + 2\,dv) + 8(-du + dv) = 5\,du + 34\,dv$.

Second method:

$h(\mathbf{U}) = 3v^2e^{2u+2v-2} + 4ve^{u+v-1}e^{-2u+2v-2} + 3ve^{u+v-1} = 3v^2e^{2u+2v-2} + 4ve^{-u+3v-3} + 3ve^{u+v-1}$;

$h_u(\mathbf{U}) = 6v^2e^{2u+2v-2} - 4ve^{-u+3v-3} + 3ve^{u+v-1}$; $h_u(\mathbf{U}_0) = 5$;

$h_v(\mathbf{U}) = (6v + 6v^2)e^{2u+2v-2} + (4 + 12v)e^{-u+3v-3} + (3 + 3v)e^{u+v-1}$; $h_v(\mathbf{U}_0) = 34$;

$d_{\mathbf{U}_0} h = h_u(\mathbf{U}_0)\, du + h_v(\mathbf{U}_0)\, dv = 5\,du + 34\,dv$.

(b) Let $\mathbf{U} = (u, v, w)$ and $\mathbf{X} = (x, y, z)$.

First method:

$\mathbf{U}_0 = (1, 1, 1)$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial u} = \dfrac{1}{u}$;

$\dfrac{\partial g_1(\mathbf{U}_0)}{\partial u} = 1$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial v} = -\dfrac{1}{v};$

$\dfrac{\partial g_1(\mathbf{U_0})}{\partial v} = -1;$

$\dfrac{\partial g_1(\mathbf{U})}{\partial w} = \dfrac{1}{w};$

$\dfrac{\partial g_1(\mathbf{U_0})}{\partial w} = 1;$

$d_{\mathbf{U_0}} g_1 = du - dv + dw;$

$\dfrac{\partial g_2(\mathbf{U})}{\partial u} = -\dfrac{2}{u};$

$\dfrac{\partial g_2(\mathbf{U_0})}{\partial u} = -2;$

$\dfrac{\partial g_2(\mathbf{U})}{\partial v} = 0;$

$\dfrac{\partial g_2(\mathbf{U_0})}{\partial v} = 0;$

$\dfrac{\partial g_2(\mathbf{U})}{\partial w} = -\dfrac{3}{w};$

$\dfrac{\partial g_2(\mathbf{U_0})}{\partial w} = -3;$

$d_{\mathbf{U_0}} g_2 = -2\,du - 3\,dw;$

$\dfrac{\partial g_3(\mathbf{U})}{\partial u} = \dfrac{1}{u};$

$\dfrac{\partial g_3(\mathbf{U_0})}{\partial u} = 1;$

$\dfrac{\partial g_3(\mathbf{U})}{\partial v} = \dfrac{1}{v};$

$\dfrac{\partial g_3(\mathbf{U_0})}{\partial v} = 1;$

$\dfrac{\partial g_3(\mathbf{U})}{\partial w} = \dfrac{2}{w};$

$\dfrac{\partial g_3(\mathbf{U_0})}{\partial w} = 2;$

$d_{\mathbf{U_0}} g_3 = du + dv + 2\,dw;$

$\mathbf{X_0} = (g_1(\mathbf{U_0}), g_2(\mathbf{U_0}), g_3(\mathbf{U_0})) = (0, 0, 0);$

$f_x(\mathbf{X}) = f_y(\mathbf{X}) = f_z(\mathbf{X}) = -e^{-(x+y+z)};$

$f_x(\mathbf{X_0}) = f_y(\mathbf{X_0}) = f_z(\mathbf{X_0}) = -1;$

$d_{\mathbf{U_0}} h = f_x(\mathbf{X_0})\, d_{\mathbf{U_0}} g_1 + f_y(\mathbf{X_0})\, d_{\mathbf{U_0}} g_2 + f_z(\mathbf{X_0})\, d_{\mathbf{U_0}} g_3 = -(du - dv + dw) - (-2\,du - 3\,dw) - (du + dv + 2\,dw) = 0.$

Second method:

$x(\mathbf{U}) + y(\mathbf{U}) + z(\mathbf{U}) = (\log u - \log v + \log w) + (-2\log u - 3\log w) + (\log u + \log v + 2\log w) = 0$; hence, $h(\mathbf{U}) \equiv 1$ and $d_{\mathbf{U}_0}h = 0$.

(b) Let $\mathbf{U} = (u, v)$ and $\mathbf{X} = (x, y)$.

First method:

$\mathbf{U}_0 = (3, \pi/2)$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial u} = \cos v$;

$\dfrac{\partial g_1(\mathbf{U}_0)}{\partial u} = 0$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial v} = -u\sin v$;

$\dfrac{\partial g_1(\mathbf{U}_0)}{\partial v} = -3$;

$d_{\mathbf{U}_0}g_1 = -3\,dv$;

$\dfrac{\partial g_2(\mathbf{U})}{\partial u} = \sin v$;

$\dfrac{\partial g_2(\mathbf{U}_0)}{\partial u} = 1$;

$\dfrac{\partial g_2(\mathbf{U})}{\partial v} = u\cos v$;

$\dfrac{\partial g_2(\mathbf{U}_0)}{\partial v} = 0$;

$d_{\mathbf{U}_0}g_2 = du$

$\mathbf{X}_0 = (g_1(\mathbf{U}_0), g_2(\mathbf{U}_0)) = (0, 3)$;

$f_x(\mathbf{X}) = f_y(\mathbf{X}) = 2(x + y)$; $f_x(\mathbf{X}_0) = f_y(\mathbf{X}_0) = 6$;

$d_{\mathbf{U}_0}h = f_x(\mathbf{X}_0)\,d_{\mathbf{U}_0}g_1 + f_y(\mathbf{X}_0)\,d_{\mathbf{U}_0}g_2 = 6\,du - 18\,dv$.

Second method:

$h(\mathbf{U}) = u^2(\cos v + \sin v)^2 = u^2(1 + 2\sin v\cos v) = u^2(1 + \sin 2v)$;

$h_u(\mathbf{U}) = 2u(1 + \sin 2v)$; $h_u(\mathbf{U}_0) = 6$;

$h_v(\mathbf{U}) = 2u^2\cos 2v$; $h_v(\mathbf{U}_0) = -18$;

$d_{\mathbf{U}_0}h = h_u(\mathbf{U}_0)\,du + h_v(\mathbf{U}_0)\,dv = 6\,du - 18\,dv$.

(d) Let $\mathbf{U} = (u, v, w)$ and $\mathbf{X} = (x, y, z)$.

First method:

$\mathbf{U}_0 = (4, \pi/3, \pi/6)$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial u} = \cos v\sin w$

$\dfrac{\partial g_1(\mathbf{U}_0)}{\partial u} = \dfrac{1}{4}$;

$\dfrac{\partial g_1(\mathbf{U})}{\partial v} = -u\sin v\sin w$;

$$\frac{\partial g_1(\mathbf{U_0})}{\partial v} = -\sqrt{3};$$
$$\frac{\partial g_1(\mathbf{U})}{\partial w} = u\cos v\cos w;$$
$$\frac{\partial g_1(\mathbf{U_0})}{\partial w} = \sqrt{3};$$
$$d_{\mathbf{U_0}}g_1 = \frac{1}{4}\,du - \sqrt{3}\,dv + \sqrt{3}\,dw;$$
$$\frac{\partial g_2(\mathbf{U})}{\partial u} = \cos v\cos w;$$
$$\frac{\partial g_2(\mathbf{U_0})}{\partial u} = \frac{\sqrt{3}}{4};$$
$$\frac{\partial g_2(\mathbf{U})}{\partial v} = -u\sin v\cos w;$$
$$\frac{\partial g_2(\mathbf{U_0})}{\partial v} = -3;$$
$$\frac{\partial g_2(\mathbf{U})}{\partial w} = -u\cos v\sin w;$$
$$\frac{\partial g_2(\mathbf{U_0})}{\partial w} = -1;$$
$$d_{\mathbf{U_0}}g_2 = \frac{\sqrt{3}}{4}\,du - 3\,dv - \,dw;$$
$$\frac{\partial g_3(\mathbf{U})}{\partial u} = \sin v$$
$$\frac{\partial g_3(\mathbf{U_0})}{\partial u} = \frac{\sqrt{3}}{2};$$
$$\frac{\partial g_3(\mathbf{U})}{\partial v} = u\cos v;$$
$$\frac{\partial g_3(\mathbf{U_0})}{\partial v} = 2;$$
$$\frac{\partial g_3(\mathbf{U})}{\partial w} = 0;$$
$$\frac{\partial g_3(\mathbf{U_0})}{\partial w} = 0;$$
$$d_{\mathbf{U_0}}g_3 = \frac{\sqrt{3}}{2}\,du + 2\,dv;$$
$$\mathbf{X_0} = (g_1(\mathbf{U_0}), g_2(\mathbf{U_0}), g_3(\mathbf{U_0})) = (1, \sqrt{3}, 2\sqrt{3});$$
$$f_x(\mathbf{X}) = 2x;\ f_x(\mathbf{X_0}) = 2;$$
$$f_y(\mathbf{X}) = 2y;\ f_y(\mathbf{X_0}) = 2\sqrt{3};$$
$$f_z(\mathbf{X}) = 2z;\ f_z(\mathbf{X_0}) = 4\sqrt{3};$$

$d_{\mathbf{U}_0}h = f_x(\mathbf{X}_0)\, d_{\mathbf{U}_0}g_1 + f_y(\mathbf{X}_0)\, d_{\mathbf{U}_0}g_2 + f_z(\mathbf{X}_0)\, d_{\mathbf{U}_0}g_3 = 2\left(\frac{1}{4}\,du - \sqrt{3}\,dv + \sqrt{3}\,dw\right) + 2\sqrt{3}\left(\frac{\sqrt{3}}{4}\,du - 3\,dv - dw\right) + 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\,du + 2\,dv\right) = 8\,du.$

Second method:

$h(\mathbf{U}) = u^2\cos^2 v\sin^2 w + u^2\cos^2 v\cos^2 w + u^2\sin^2 v = u^2$; $h_u(\mathbf{U}) = 2u$; $h_v(\mathbf{U}) = 0$; $h_w(\mathbf{U}) = 0$; $h_u(\mathbf{U}_0) = 8$; $h_v(\mathbf{U}_0) = 0$; $h_w(\mathbf{U}_0) = 0$; $dh = 8\,du$.

5.4.3. $h_r = f_x x_r + f_y y_r + f_z z_r = f_x\cos\theta + f_y\sin\theta$;

$h_\theta = f_x x_\theta + f_y y_\theta + f_z z_\theta = f_x(-r\sin\theta) + f_y(r\cos\theta) = r(-f_x\sin\theta + f_y\cos\theta)$;

$h_r = f_x x_z + f_y y_z + f_z = f_z$.

5.4.4. $h_r = f_x x_r + f_y y_r + f_z z_r = f_x\sin\phi\cos\theta + f_y\sin\phi\sin\theta + f_z\cos\phi$;

$h_\theta = f_x x_\theta + f_y y_\theta + f_z z_\theta = r\sin\phi(-f_x\sin\theta + f_y\cos\theta)$;

$h_\phi = f_x x_\phi + f_y y_\phi + f_z z_\phi = r(f_x\cos\phi\cos\theta + f_y\cos\phi\sin\theta - f_z\sin\phi$.

5.4.5. **(a)** $h_u = 2uf'$; $h_v = 2vf'$; $vh_u - uh_v = 2(vu - uv)f' = 0$.

(b) $h_u = f'\cos u$; $h_v = -f'\sin v$; $h_u\sin v + h_v\cos u = (\sin v\cos u - \cos u\sin v)f' = 0$.

(c) $h_u = \dfrac{1}{v}f'$; $h_v = -\dfrac{u}{v^2}f'$; $uh_u + vh_v = \left(\dfrac{u}{v} - \dfrac{u}{v}\right)f' = 0$.

(d) $h_u = (f_x - f_y)g_u$; $h_v = (f_x - f_y)g_v$; $dh = h_u\,du + h_v\,dv = (f_x - f_y)(g_u\,du + g_v\,dv) = (f_x - f_y)\,dg$.

5.4.6. $h_y = g_x x_y + g_y + g_w w_y$; $h_z = g_x x_z + g_z + g_w w_z$.

5.4.7. Let $F(u,v) = \int_u^v f(t)\,dt$, which can be rewritten as $F(u,v) = \displaystyle\int_{t_0}^{v} f(t)\,dt - \int_{t_0}^{u} f(t)\,dt$; From Theorem 3.3.11, $F_u(u,v) = -f(u)$ and $F_v(u,v) = f(v)$. Let $h(x) = \displaystyle\int_{u(x)}^{v(x)} f(t)\,dt = F(u(x),v(x))$; from the chain rule, $h'(x) = F_v(u(x),v(x))v'(x) - F_u(u(x),v(x))u'(x) = f(v(x))v'(x) - f(u(x))u'(x)$.

5.4.8. For a fixed $(x_1,x_2,\dots,x_n)$, differentiating the identity $f(tx_1,tx_2,\dots,tx_n) = t^r f(x_1,x_2,\dots,x_n)$ with respect to t and using the chain rule on the left yields

$$\sum_{i=1}^{n} x_i f_{x_i}(tx_1,tx_2,\dots,tx_n) = rt^{r-1}f(x_1,x_2,\dots,x_n).$$

Now set $t = 1$.

5.4.9. $f_x = h_r r_x + h_\theta\theta_x$;

$f_y = h_r r_y + h_\theta\theta_y$;

$f_{xx} = h_r r_{xx} + h_\theta\theta_{xx} + r_x(h_{rr}r_x + h_{r\theta}\theta_x) + \theta_x(h_{\theta r}r_x + h_{\theta\theta}\theta_x)$;

$f_{yy} = h_r r_{yy} + h_\theta\theta_{yy} + r_y(h_{rr}r_y + h_{r\theta}\theta_y) + \theta_y(h_{\theta r}r_y + h_{\theta\theta}\theta_y)$;

$f_{xx} + f_{yy} = h_r(r_{xx} + r_{yy}) + h_\theta(\theta_{xx} + \theta_{yy}) + h_{rr}(r_x^2 + r_y^2) + h_{\theta\theta}(\theta_x^2 + \theta_y^2) + 2h_{r\theta}(\theta_x r_x + \theta_y r_y)$;

Now we evaluate the multipliers of h_r, h_θ, h_{rr}, $h_{\theta\theta}$ and $h_{r\theta} = h_{\theta,r}$ on the right.

$r_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r} = \cos\theta$; $r_y = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{r} = \sin\theta$;

$r_x^2 + r_y^2 = 1$;

$\theta_x = \dfrac{1}{1+(y/x)^2}\left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2+y^2} = -\dfrac{\sin\theta}{r}$;

$\theta_y = \dfrac{1}{1+(y/x)^2}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}$;

$\theta_x^2 + \theta_y^2 = \dfrac{1}{r^2}$;

$\theta_x r_x + \theta_y r_y = 0$;

$r_{xx} = -\theta_x \sin\theta = \dfrac{\sin^2\theta}{r}$; $r_{yy} = \theta_y \cos\theta = \dfrac{\cos^2\theta}{r}$

$r_{xx} + r_{yy} = \dfrac{1}{r}$;

$\theta_{xx} = -\dfrac{\theta_x \cos\theta}{r} + \dfrac{r_x \sin\theta}{r^2} = \dfrac{2\sin\theta\cos\theta}{r^2}$;

$\theta_{yy} = -\dfrac{\theta_y \sin\theta}{r} + \dfrac{r_y \cos\theta}{r^2} = -\dfrac{2\sin\theta\cos\theta}{r^2}$;

$\theta_{xx} + \theta_{yy} = 0$.

5.4.10. $h_u = f_x a_u + f_y b_u$;

$h_{uu} = f_x a_{uu} + f_y b_{uu} + a_u(f_{xx}a_u + f_{xy}b_u) + b_u(f_{yx}a_u + f_{yy}b_u)$;

$h_v = f_x a_v + f_y b_v$;

$h_{vv} = f_x a_{vv} + f_y b_{vv} + a_v(f_{xx}a_v + f_{xy}b_v) + b_v(f_{yx}a_v + f_{yy}b_v)$;

$h_{uu} + h_{vv} = f_x(a_{uu} + a_{vv}) + f_y(b_{uu} + b_{vv}) + (a_u^2 + a_v^2)f_{xx} + (b_u^2 + b_v^2)f_{yy} + (a_u b_u + a_v b_v)(f_{xy} + f_{yx})$.

Since $a_u = b_v$ and $a_v = -b_u$, $a_u b_u + a_v b_v = 0$ and $b_u^2 + b_v^2 = a_u^2 + a_v^2$. Differentiating $a_u = b_v$ and $a_v = -b_u$ with respect to u and v respectively yields $a_{uu} = b_{vu}$ and $a_{vv} = -b_{uv}$, so $a_{uu} + a_{vv} = b_{vu} - b_{uv} = 0$ (Theorem 5.3.3). Differentiating $a_u = b_v$ and $a_v = -b_u$ with respect to v and u respectively yields $a_{uv} = b_{vv}$ and $a_{vu} = -b_{uu}$, so $b_{uu} + b_{vv} = -a_{vu} + a_{uv} = 0$ (Theorem 5.3.3). Therefore, $h_{uu} + h_{vv} = (f_{xx} + f_{yy})(a_u^2 + a_v^2)$.

5.4.11. $u_x = f' + g'$; $u_{xx} = f'' + g''$; $u_t = -cf' + cg'$; $u_{tt} = c^2 f'' + c^2 g = c^2 u''_{xx}$.

5.4.12. **(a)** $h_u = f_x + f_y$; $h_{uv} = (f_{xx} - f_{xy}) + (f_{yx} - f_{yy}) = f_{xx} - f_{yy}$ (Theorem 5.3.3).

(b) $h_u = f_x + f_y$ $h_{uu} = (f_{xx} + f_{xy}) + (f_{yx} + f_{yy})$;

$h_v = f_x - f_y$ $h_{vv} = (f_{xx} - f_{xy}) - (f_{yx} - f_{yy})$;

$h_{uu} + h_{vv} = 2(f_{xx} + f_{yy})$

5.4.13. From Exercise 5.4.4, $h_r = f_x \sin\phi\cos\theta + f_y \sin\phi\sin\theta + f_z\cos\phi$;

$$\begin{aligned}
h_{rr} &= (f_{xx}x_r + f_{xy}y_r + f_{xz}z_r)\sin\phi\cos\theta + (f_{yx}x_r + f_{yy}y_r + f_{yz}z_r)\sin\phi\sin\theta \\
&\quad + (f_{zx}x_r + f_{zy}y_r + f_{zz}z_r)\cos\phi \\
&= f_{xx}\sin^2\phi\cos^2\theta + f_{yy}\sin^2\phi\sin^2\theta + f_{zz}\cos^2\phi \\
&\quad + f_{xy}(y_r\sin\phi\cos\theta + x_r\sin\phi\sin\theta) + f_{yz}(z_r\sin\phi\sin\theta + y_r\cos\phi) \\
&\quad + f_{xz}(z_r\sin\phi\cos\theta + x_r\cos\phi) \\
&= 2f_{xy}(\sin^2\phi\sin\theta\cos\theta) + 2f_{yz}(\sin\phi\cos\phi\sin\theta) + 2f_{xz}(\sin\phi\cos\phi\cos\theta) \\
&= f_{xx}\sin^2\phi\cos^2\theta + f_{yy}\sin^2\phi\sin^2\theta + f_{zz}\cos^2\phi \\
&\quad + f_{xy}\sin^2\phi\sin 2\theta + f_{yz}\sin 2\phi\sin\theta + f_{xz}\sin 2\phi\cos\theta;
\end{aligned}$$

$$\begin{aligned}
h_{r\theta} &= (-f_x\sin\theta + f_y\cos\theta)\sin\phi + (f_{xx}x_\theta + f_{xy}y_\theta)\sin\phi\cos\theta \\
&\quad + (f_{yx}x_\theta + f_{yy}y_\theta)\sin\phi\sin\theta + (f_{zx}x_\theta + f_{zy}y_\theta)\cos\phi \\
&= (-f_x\sin\theta + f_y\cos\theta)\sin\phi + r(f_{yy} - f_{xx})\sin^2\phi\sin\theta\cos\theta \\
&\quad + rf_{xy}\sin^2\phi(\cos^2\theta - \sin^2\theta) + r(f_{zy}\cos\theta - f_{zx}\sin\theta)\sin\phi\cos\phi \\
&= (-f_x\sin\theta + f_y\cos\theta)\sin\phi + \frac{r}{2}(f_{yy} - f_{xx})\sin^2\phi\sin 2\theta \\
&\quad + rf_{xy}\sin^2\phi\cos 2\theta + \frac{r}{2}(f_{zy}\cos\theta - f_{zx}\sin\theta)\sin 2\phi
\end{aligned}$$

5.4.14. If $h_{uv} = 0$ for all (u, v), then h_u is independent of v, by Theorem 2.3.12. Therefore, $h_u(u, v) = U_0(u)$ and $h(u, v) = U_1(u) + V_1(v)$, where $U_1'(u) = U_0(u)$. If $f_{xx} - f_{yy} = 0$, Exercise 5.4.12**(a)** and this result imply that $f(u + v, u - v) = U_1(u) + V_1(v)$. Setting $x = u + v$ and $y = u - v$ yields

$$f(x, y) = U_1\left(\frac{x+y}{2}\right) + V_1\left(\frac{x-y}{2}\right),$$

and the stated result follows, with $U(u) = U_1(u/2)$ and $V(v) = V_1(v/2)$.

5.4.15. False; let D be the entire xy-plane except fior the nonnegative y axis, and

$$f(x, y) = \begin{cases} 0 & \text{if } y < 0, \\ y^3 & \text{if } y \ge 0 \text{ and } x < 0, \\ y^4 & \text{if } y \ge 0 \text{ and } x > 0. \end{cases}$$

Then f is differentiable and $f_x = 0$ on D, but $f(x, y) \ne f(-x, y)$ if $y > 0$ and $x \ne 0$.

5.4.16. In **(a)** and **(b)**,

$$\begin{aligned}
T_3(x, y) &= f(0,0) + (f_x(0,0)x + f_y(0,0)y) \\
&\quad + \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2) \\
&\quad + \frac{1}{6}(f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3);
\end{aligned}$$

(a) $f(x,y) = e^x \cos y$; $f(0,0) = 1$;
$f_x(x,y) = e^x \cos y$; $f_x(0,0) = 1$;
$f_y(x,y) = -e^x \sin y$; $f_y(0,0) = 0$;
$f_{xx}(x,y) = e^x \cos y$; $f_{xx}(0,0) = 1$;
$f_{xy}(x,y) = -e^x \sin y$; $f_{xy}(0,0) = 0$;
$f_{yy}(x,y) = -e^x \cos y$; $f_{yy}(0,0) = -1$;
$f_{xxx}(x,y) = e^x \cos y$; $f_{xxx}(0,0) = 1$;
$f_{xxy}(x,y) = -e^x \sin y$; $f_{xxy}(0,0) = 0$;
$f_{xyy}(x,y) = -e^x \cos y$; $f_{xyy}(0,0) = -1$;
$f_{yyy}(x,y) = e^x \sin y$; $f_{yyy}(0,0) = 0$;
$T_3(x,y) = 1 + x + \dfrac{x^2 - y^2}{2} + \dfrac{x^3}{6} - \dfrac{xy^2}{2}$.
(b) $f(x,y) = e^{-x-y}$; $f(0,0) = 1$;
$f_x(x,y) = -e^{-x-y}$; $f_x(0,0) = -1$;
$f_y(x,y) = -e^{-x-y}$; $f_y(0,0) - 1$;
$f_{xx}(x,y) = e^{-x-y}$; $f_{xx}(0,0) = 1$;
$f_{xy}(x,y) = e^{-x-y}$; $f_{xy}(0,0) = 1$;
$f_{yy}(x,y) = e^{-x-y}$; $f_{yy}(0,0) = 1$;
$f_{xxx}(x,y) = -e^{-x-y}$; $f_{xxx}(0,0) = -1$;
$f_{xxy}(x,y) = -e^{-x-y}$; $f_{xxy}(0,0) = -1$;
$f_{xyy}(x,y) = -e^{-x-y}$; $f_{xyy}(0,0) = -1$;
$f_{yyy}(x,y) = -e^{-x-y}$; $f_{yyy}(0,0) = -1$;
$T_3(x,y) = 1 - x - y + \dfrac{x^2}{2} + xy + \dfrac{y^2}{2} - \dfrac{x^3}{6} - \dfrac{x^2y}{2} - \dfrac{xy^2}{2} - \dfrac{y^3}{6}$.
(c) If $r_1 + r_2 + r_3 \le 3$, then

$$\frac{\partial^{r_1+r_2+r_3} f(x,y,z)}{\partial x^{r_2} \partial x^{r_2} \partial x^{r_3}} = k(x + y + z - 3)^{5-r+1+r_2+r_3},$$

where k is a constant. Since the right side is zero if $(x,y,z) = (1,1,1)$, $T_3(x,y,z) = 0$.
(d) All partial derivatives of f through the third order equal zero at $(0,0,0)$ except for $f_{xyz}(0,0,0) = \cos x \cos y \cos z$, which equals one at $(0,0,0)$. Therefore,

$$T_3(x,y,z) = \frac{1}{3!} \frac{3!}{1!1!1!} f_{xyz}(0,0,0)xyz = xyz.$$

5.4.17. Let $\mathbf{X}_0 = (x_{10}, x_{20}, \dots, x_{n0})$. From Eqn. (5.4.23), the left side of Eqn. (5.4.35) can be written as

$$\left|\sum_{i_1,i_2,\dots,i_k=1}^{n}\left(\frac{\partial^k f(\widetilde{\mathbf{X}})}{\partial x_{i_k}\partial x_{i_{k-1}}\cdots\partial x_{i_1}}-\frac{\partial^k f(\mathbf{X}_0)}{\partial x_{i_k}\partial x_{i_{k-1}}\cdots\partial x_{i_1}}\right)\right.$$
$$\left.\times(x_{i_1}-x_{i_10})(x_{i_2}-x_{i_20})\cdots(x_{i_k}-x_{i_k0})\right|.$$

From Eqn. (5.4.32), each term in this sum is less in magnitude than $\epsilon|\mathbf{X}-\mathbf{X}_0|^k$, and there are n^k terms. This yields Eqn. (5.4.35).

5.4.18. Theorem 5.4.9 with $n=1$ requires that $f^{(k)}$ be continuous at x_0. Theorem 2.5.1 requires only that $f^{(k)}(x_0)$ exist.

5.4.19. By Theorem 5.2.13, p assumes a minimum value $\rho>0$ on $\{Y \mid |Y|=1\}$. If $\mathbf{X}\ne 0$, then $p(\mathbf{X}/|\mathbf{X}|)\ge\rho$, and the homogeneity implies that $p(\mathbf{X})\ge\rho|\mathbf{X}|^r$, which also obviously holds if $x=0$. This implies Eqn. (5.4.41), since $d_{\mathbf{X}_0}^{(k)}f$ is homogeneous of degree k.

Suppose that $d_{\mathbf{X}_0}^{(k)}f$ is negative definite. By an argument similar to the solution of Exercise 5.4.19, it can be shown that there is a $\rho>0$ such that

$$\frac{(d_{\mathbf{X}_0}^{(k)}f)(\mathbf{X}-\mathbf{X}_0)}{k!}\le-\rho|\mathbf{X}-\mathbf{X}_0|^k \tag{A}$$

for all $\mathbf{X}$. Since

$$\lim_{\mathbf{X}\to\mathbf{X}_0}\frac{f(\mathbf{X})-f(\mathbf{X}_0)-\dfrac{1}{k!}(d_{\mathbf{X}_0}^{(k)})(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|^k}=0,$$

there is a $\delta>0$ such that

$$\frac{f(\mathbf{X})-f(\mathbf{X}_0)-\dfrac{1}{k!}(d_{\mathbf{X}_0}^{(k)}f)(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|^k}<\frac{\rho}{2}\quad\text{if}\quad|\mathbf{X}-\mathbf{X}_0|<\delta. \tag{A}$$

Therefore,

$$f(\mathbf{X})-f(\mathbf{X}_0)<\frac{1}{k!}(d_{\mathbf{X}_0}^{(k)})(\mathbf{X}-\mathbf{X}_0)+\frac{\rho}{2}|\mathbf{X}-\mathbf{X}_0|^k\quad\text{if}\quad|\mathbf{X}-\mathbf{X}_0|<\delta.$$

This and (A) imply that

$$f(\mathbf{X})-f(\mathbf{X}_0)<-\frac{\rho}{2}|\mathbf{X}-\mathbf{X}_0|^k\quad\text{if}\quad|\mathbf{X}-\mathbf{X}_0|<\delta,$$

which implies that $\mathbf{X}_0$ is a local maximum point of f.

5.4.21. $p(x,y)=x^2-2xy+y^2+x^4+y^4$; $p_x(x,y)=2x-2y+4x^3$; $p_y(x,y)=-2x+2y+4y^3$. Since $p_x(0,0)=p_y(0,0)=0$, $(0,0)$ is a critical point of p. $p_{xx}(x,y)=2+12x^2$; $p_{xy}(x,y)=-2$; $p_{yy}(x,y)=2+12y^2$; $(d_{(0,0)}^{(2)}p)(x,y)=2x^2-4xy+2y^2=2(x-y)^2$ is positive semidefinite.

$q(x, y) = x^2-2xy+y^2-x^4-y^4$. $q_x(x, y) = 2x-2y-4x^3$; $q_y(x, y) = -2x+2y-4y^3$. Since $q_x(0,0) = q_y(0,0) = 0$, $(0,0)$ is a critical point of q. $q_{xx}(x, y) = 2 - 12x^2$; $q_{xy}(x, y) = -2$; $q_{yy}(x, y) = 2-12y^2$; $(d^{(2)}_{(0,0)}q)(x, y) = 2x^2-4xy+2y^2 = 2(x-y)^2$ is positive semidefinite.

(b) $D_p = p_{xx}(0,0)p_{xy}(0,0) - p^2_{xy}(0,0) = 2\cdot 2-2^2 = 0$ and $D_q = q_{xx}(0,0)q_{xy}(0,0) - q^2_{xy}(0,0) = 2\cdot 2-2^2 = 0$.

(c) $p(x, y) = (x-y)^2 + x^4 + y^4 \geq 0$ for all (x, y). Since $p(0,0) = 0$, $(0,0)$ is a local minimum point of p.

$q(x, y) = (x-y)^2 - x^4 - y^4$, so $q(x, x) = -2x^4 < 0$ if $x \neq 0$. Also, $q(x, 0) = x^2(1-x^2) > 0$ if $0 < x < 1$. Since $q(0,0) = 0$, $(0,0)$ is not a local extreme point of q.

CHAPTER 6

VECTOR-VALUED FUNCTIONS OF SEVERAL VARIABLES

6.1 LINEAR TRANSFORMATIONS AND MATRICES

6.1.1. Recall that (A) $\mathbf{L}(\mathbf{U}+\mathbf{V}) = \mathbf{U}+\mathbf{V}$ and (B) $\mathbf{L}(a\mathbf{U}) = a\mathbf{L}(\mathbf{U})$ if $\mathbf{U}$ and $\mathbf{V}$ are vectors and a is a scalar. From (A) with $\mathbf{U} = a_1\mathbf{X}_1$ and $v = a_2\mathbf{X}_2$, $\mathbf{L}(a_1\mathbf{X}_1 + a_2\mathbf{X}_2) = \mathbf{L}(a_1\mathbf{X}_1) + \mathbf{L}(a_2\mathbf{X}_2)$, so (B) implies that $\mathbf{L}(a_1\mathbf{X}_1 + a_2\mathbf{X}_2) = a_1\mathbf{L}(\mathbf{X}_1) + a_2\mathbf{L}(\mathbf{X}_2)$. This proves the proposition with $k = 2$. Now suppose that $k > 2$ and the proposition is true with k replaced by $k-1$. Then (A) with $\mathbf{U} = a_1\mathbf{X}_1 + \cdots + a_{k-1}\mathbf{X}_{k-1}$ and $\mathbf{V} = a_k\mathbf{X}_k$ yields

$$\begin{aligned}
\mathbf{L}(a_1\mathbf{X}_1 + \cdots + a_{k-1}\mathbf{X}_{k-1} + a_k\mathbf{X}_k) &= \mathbf{L}((a_1\mathbf{X}_1 + \cdots + a_{k-1}\mathbf{X}_{k-1}) + a_k\mathbf{X}_k) \\
&= \mathbf{L}(a_1\mathbf{X}_1 + \cdots + a_{k-1}\mathbf{X}_{k-1}) + \mathbf{L}(a_k\mathbf{X}_k) \\
&= (a_1\mathbf{L}(\mathbf{X}_1) + \cdots + a_{k-1}\mathbf{L}(\mathbf{X}_{k-1})) + a_k\mathbf{L}(\mathbf{X}_k)
\end{aligned}$$

by the induction assumption and (B). This completes the induction.

6.1.2.

$$\mathbf{L}(\mathbf{X}+\mathbf{Y}) = \begin{bmatrix} a_{11}(x_1+y_1)+a_{12}(x_2+y_2)+\cdots+a_{1n}(x_n+y_n) \\ a_{21}(x_1+y_1)+a_{22}(x_2+y_2)+\cdots+a_{2n}(x_n+y_n) \\ \vdots \\ a_{m1}(x_1+y_1)+a_{m2}(x_2+y_2)+\cdots+a_{mn}(x_n+y_n) \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{bmatrix} + \begin{bmatrix} a_{11}y_1+a_{12}y_2+\cdots+a_{1n}y_n \\ a_{21}y_1+a_{22}y_2+\cdots+a_{2n}y_n \\ \vdots \\ a_{m1}y_1+a_{m2}y_2+\cdots+a_{mn}y_n \end{bmatrix}$$
$$= \mathbf{L}(\mathbf{X})+\mathbf{L}(\mathbf{Y}).$$

$$\mathbf{L}(a\mathbf{X}) = \begin{bmatrix} a_{11}(ax_1)+a_{12}(ax_2)+\cdots+a_{1n}(ax_n) \\ a_{21}(ax_1)+a_{22}(ax_2)+\cdots+a_{2n}(ax_n) \\ \vdots \\ a_{m1}(ax_1)+a_{m2}(ax_2)+\cdots+a_{mn}(ax_n) \end{bmatrix}$$
$$= a\begin{bmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{bmatrix} = a\mathbf{L}(\mathbf{X}).$$

6.1.7. $(\mathbf{A}+\mathbf{B})+\mathbf{C} = [(a_{ij}+b_{ij})+c_{ij}] = [a_{ij}+(b_{ij}+c_{ij})] = \mathbf{A}+(\mathbf{B}+\mathbf{C})$.

6.1.8. **(a)** $r(s\mathbf{A}) = r[sa_{ij}] = [r(sa_{ij})] = [(rs)a_{ij}] = (rs)\mathbf{A}$.

(b) $(r+s)\mathbf{A} = [(r+s)a_{ij}] = [ra_{ij}+sa_{ij}] = r\mathbf{A}+s\mathbf{A}$

(c) $r(\mathbf{A}+\mathbf{B}) = r[a_{ij}+b_{ij}] = [r(a_{ij}+b_{ij})] = [ra_{ij}+rb_{ij}] = r\mathbf{A}+r\mathbf{B}$.

6.1.9. Let $\mathbf{D} = [d_{ij}] = \mathbf{AB}$ and $F = [f_{ij}] = \mathbf{BC}$. Wemust show that $\mathbf{DC} = \mathbf{AF}$. By definition, $d_{ik} = \sum_{r=1}^{p} a_{ir}b_{rk}$ $(1 \le i \le m, 1 \le k \le q)$, and the (i,j)th entry of $\mathbf{DC}$ is $\sum_{k=1}^{q} d_{ik}c_{kj} = \sum_{k=1}^{q}\left(\sum_{r=1}^{p} a_{ir}b_{rk}\right)c_{kj}$ $(1 \le i \le m, 1 \le j \le n)$. Changing the order of summation shows that the sum on the right equals $\sum_{r=1}^{p} a_{ir}\sum_{k=1}^{q} b_{rk}c_{kj} = \sum_{r=1}^{p} a_{ir}f_{rj}$, which is the (i,j)th entry of $\mathbf{AF}$. Therefore, $\mathbf{DC} = \mathbf{AF}$.

6.1.10. If $\mathbf{A}+\mathbf{B}$ is defined, then $\mathbf{A}$ and $\mathbf{B}$ have the same number m of rows and the same number n of columns. If $\mathbf{AB}$ is defined, then the number n of columns of $\mathbf{A}$ must equal the number m of rows of $\mathbf{B}$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order.

6.1.11. **(a)** From the definition of matrrix multiplication,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

(b) $(c_1\mathbf{L}_1 + c_2\mathbf{L}_2)(\mathbf{X}) = c_1\mathbf{L}_1(\mathbf{X}) + c_2\mathbf{L}_2(\mathbf{X}) = c_1\mathbf{A}_1\mathbf{X} + c_2\mathbf{A}_2\mathbf{X} = (c_1\mathbf{A}_1 + c_2\mathbf{A}_2)\mathbf{X}$.
(c) $\mathbf{L}_3(\mathbf{X}) = \mathbf{L}_2(\mathbf{L}_1(\mathbf{X})) = \mathbf{A}_2(\mathbf{L}_1(\mathbf{X})) = \mathbf{A}_2(\mathbf{A}_1\mathbf{X}) = (\mathbf{A}_2\mathbf{A}_1)\mathbf{X}$ from Exercise 6.1.9.

6.1.16. If $\mathbf{Y} = \mathbf{AX}$, then $|\mathbf{Y}|^2 = \displaystyle\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2 \le \left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^2\right)\right)|\mathbf{X}|^2$, by Schwarz's inequality. Therefore, $|\mathbf{Y}|^2 \le mn\lambda^2|\mathbf{X}|^2$, so $\|\mathbf{A}\| \le \lambda\sqrt{mn}$.

6.1.17. If $\|\mathbf{A}\| = 0$, then $\mathbf{AX} = 0$ for every $\mathbf{X}$, which implies that $\mathbf{A} = 0$.

6.1.18. $|(\mathbf{A}+\mathbf{B})\mathbf{X}| \le |\mathbf{AX}| + |\mathbf{BX}| \le (\|\mathbf{A}\| + \|\mathbf{B}\|)|\mathbf{X}|$; hence, $\|\mathbf{A}+\mathbf{B}\| \le \|\mathbf{A}\| + \|\mathbf{B}\|$.

6.1.19. $|(\mathbf{AB})\mathbf{X}| = |\mathbf{A}(\mathbf{BX})| \le \|\mathbf{A}\||\mathbf{BX}| \le \|\mathbf{A}\|(\|\mathbf{B}\||\mathbf{X}|)$; hence, $\|\mathbf{AB}\| \le \|\mathbf{A}\|\|\mathbf{B}\|$.

6.1.20. **(a)** The matrix of the system is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & -3 \end{bmatrix}$; $\det(A) = 6$. By Cramer's rule, $x = \dfrac{1}{6}\begin{vmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 2 & -2 & -3 \end{vmatrix} = \dfrac{12}{6} = 2$; $y = \dfrac{1}{6}\begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \dfrac{18}{6} = 3$; $z = \dfrac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -2 & 2 \end{bmatrix} = -\dfrac{12}{6} = -2$.

(b) $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -2 & 2 \\ 4 & 2 & -3 \end{bmatrix}$; $\det(A) = 5$. By Cramer's rule, $x = \dfrac{1}{5}\begin{vmatrix} 5 & 1 & -1 \\ 0 & -2 & 2 \\ 14 & 2 & -3 \end{vmatrix} = \dfrac{10}{5} = 2$; $y = \dfrac{1}{5}\begin{vmatrix} 1 & 5 & -1 \\ 3 & 0 & 2 \\ 4 & 14 & -3 \end{vmatrix} = \dfrac{15}{5} = 3$; $z = \dfrac{1}{5}\begin{vmatrix} 1 & 1 & 5 \\ 3 & -2 & 0 \\ 4 & 2 & 14 \end{vmatrix} = \dfrac{0}{5} = 0$.

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$; $\det(A) = -2$. By Cramer's rule,

$x = -\dfrac{1}{2}\begin{vmatrix} -5 & 2 & 3 \\ -1 & 0 & -1 \\ -4 & 1 & 2 \end{vmatrix} = -\dfrac{4}{2} = -2$; $y = -\dfrac{1}{2}\begin{vmatrix} 1 & -5 & 3 \\ 1 & -1 & -1 \\ 1 & -4 & 2 \end{vmatrix} = -\dfrac{0}{2} = 0$; $z = -\dfrac{1}{2}\begin{vmatrix} 1 & 2 & -5 \\ 1 & 0 & -1 \\ 1 & 1 & -4 \end{vmatrix} = -\dfrac{2}{2} = -1$.

(d) $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -2 \\ 2 & 1 & -3 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & -1 & 0 \end{bmatrix}$; $\det(A) = -20$. By Cramer's rule,

$x = -\frac{1}{20}\begin{bmatrix} 1 & -1 & 1 & -2 \\ 4 & 1 & -3 & 3 \\ 13 & 2 & 0 & 1 \\ 4 & 1 & -1 & 0 \end{bmatrix} = \frac{60}{20} = 3$; $y = -\frac{1}{20}\begin{bmatrix} 1 & 1 & 1 & -2 \\ 2 & 4 & -3 & 3 \\ 3 & 13 & 0 & 1 \\ 2 & 4 & -1 & 0 \end{bmatrix} = \frac{20}{20} = 1$; $z = -\frac{1}{20}\begin{bmatrix} 1 & -1 & 1 & -2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 13 & 1 \\ 2 & 1 & 4 & 0 \end{bmatrix} = \frac{60}{20} = 3$; $w = -\frac{1}{20}\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -3 & 4 \\ 3 & 2 & 0 & 13 \\ 2 & 1 & -1 & 4 \end{bmatrix} = \frac{40}{20} = 2$.

6.1.21. **(a)** $\det(\mathbf{A}) = \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = 10$; $\mathbf{C} = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$; $\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$.

(b) $\det\mathbf{A} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{vmatrix} = -2$; $c_{11} = \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 1$; $c_{12} = -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = -3$; $c_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$; $c_{21} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$; $c_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$; $c_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$; $c_{31} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = -2$; $c_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = 4$; $c_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$; $\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 2 \\ 3 & 1 & -4 \\ -1 & -1 & 2 \end{bmatrix}$.

(c) $\det\mathbf{A} = \begin{vmatrix} 4 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = -25$; $c_{11} = \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = -4$; $c_{12} = -\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = -6$; $c_{13} = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3$; $c_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$; $c_{22} = \begin{vmatrix} 4 & 1 \\ 0 & 2 \end{vmatrix} = 8$; $c_{23} = -\begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix} = -4$; $c_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$; $c_{32} = -\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = -5$; $c_{33} = \begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix} = -10$; $\mathbf{A}^{-1} = \frac{1}{25}\begin{bmatrix} 4 & 3 & -5 \\ 6 & -8 & 5 \\ -3 & 4 & 10 \end{bmatrix}$.

(d) $\det\mathbf{A} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2$; $c_{11} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$; $c_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$; $c_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$; $c_{21} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$; $c_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$; $c_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$; $c_{31} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$; $c_{32} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$; $c_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $\mathbf{A}^{-1} =$

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

(e) $\det \mathbf{A} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 49$; $c_{11} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -1 & 2 \end{vmatrix} = 21$;
$c_{12} = -\begin{vmatrix} -2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -1 & 2 \end{vmatrix} = 14$; $c_{13} = \begin{vmatrix} -2 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 0$; $c_{14} = -\begin{vmatrix} -2 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{vmatrix} =$
0; $c_{21} = -\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -1 & 2 \end{vmatrix} = -14$; $c_{22} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -1 & 2 \end{vmatrix} = 7$; $c_{23} = -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 0$;
$c_{24} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{vmatrix} = 0$; $c_{31} = \begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix} = 0$; $c_{32} = -\begin{vmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix} = 0$;
$c_{33} = \begin{vmatrix} 1 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 14$; $c_{34} = -\begin{vmatrix} 1 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 7$; $c_{41} = -\begin{vmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 3 \end{vmatrix} = 0$;
$c_{42} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 2 & 3 \end{vmatrix} = 0$; $c_{43} = -\begin{vmatrix} 1 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -21$; $c_{44} = \begin{vmatrix} 1 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 14$;

$$\mathbf{A}^{-1} = \frac{1}{7}\begin{bmatrix} 3 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(f) $\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 2 & 2 & -1 & 3 \\ -1 & 4 & 1 & 2 \\ 3 & 1 & 0 & 1 \end{vmatrix} = -10$; $c_{11} = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1$; $c_{12} = -\begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} =$
14; $c_{13} = \begin{vmatrix} 2 & 2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} = -21$; $c_{14} = -\begin{vmatrix} 2 & 2 & -1 \\ -1 & 4 & 1 \\ 3 & 1 & 0 \end{vmatrix} = -17$; $c_{21} = -\begin{vmatrix} 1 & 2 & -1 \\ 4 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} =$
2; $c_{22} = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = 18$; $c_{23} = -\begin{vmatrix} 1 & 1 & -1 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} = -22$; $c_{24} = \begin{vmatrix} 1 & 1 & 2 \\ -1 & 4 & 1 \\ 3 & 1 & 0 \end{vmatrix} =$
-24; $c_{31} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 0$; $c_{32} = -\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 0 & 1 \end{vmatrix} = -10$; $c_{33} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} =$
10; $c_{34} = -\begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & -1 \\ 3 & 1 & 0 \end{vmatrix} = 10$; $c_{41} = -\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 4 & 1 & 2 \end{vmatrix} = -5$; $c_{42} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 1 & 2 \end{vmatrix} =$

$$-20;\ c_{43} = -\begin{vmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ -1 & 4 & 2 \end{vmatrix} = 25;\ c_{44} = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & -1 \\ -1 & 4 & 1 \end{vmatrix} = 25;\ \mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} -1 & -2 & 0 & 5 \\ -14 & -18 & 10 & 20 \\ 21 & 22 & -10 & -25 \\ 17 & 24 & -10 & -25 \end{bmatrix}.$$

6.1.22. Eqn. (6.1.10) and the continuity of $\{a_{ij}\}$ imply that the entries of $\mathbf{A}^{-1}$, and therefore of $\mathbf{B}$, are continuous on K. If the conclusion were false, there would be integers r and s in $\{1, 2, \dots, m\}$, an $\epsilon_0 > 0$, and sequences $\{\mathbf{X}_j\}$ and $\{\mathbf{Y}_j\}$ in K such that (A) $\lim_{j\to\infty} |\mathbf{X}_j - \mathbf{Y}_j| = 0$, but (B) $|b_{rs}(\mathbf{X}_j, \mathbf{Y}_j)| \ge \epsilon_0$, $j \ge 1$. By the compactness of K and Exercise 5.1.3, there is a subsequence $\{\mathbf{X}_{j_k}\}$ of $\{\mathbf{X}_j\}$ which converges to a limit $\overline{\mathbf{X}}$ in K, and (A) implies that $\{\mathbf{Y}_{j_k}\}$ does also. Therefore, $\lim_{j\to\infty} (\mathbf{X}_{j_k}, \mathbf{Y}_{j_k}) = (\overline{\mathbf{X}}, \overline{\mathbf{X}})$ in R^{2m}. This implies that $\lim_{k\to\infty} b_{rs}(\mathbf{X}_{j_k}, \mathbf{Y}_{j_k}) = b_{rs}(\overline{\mathbf{X}}, \overline{\mathbf{X}}) = 0$ (Theorem 4.2.6), which contradicts (B).

6.2 CONTINUITY AND DIFFERENTIABILITY OF TRANSFORMATIONS

6.2.1. If

$$|f_i(\mathbf{X}) - f_i(\mathbf{X}_0)| < \frac{\epsilon}{\sqrt{m}},\ 1 \le i \le m,\ \text{ if } |\mathbf{X} - \mathbf{X}_0| < \delta \text{ and } \mathbf{X} \in D_{\mathbf{F}},$$

then

$$|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0)| = \left(\sum_{i=1}^{m} (f_i(\mathbf{X}) - f_i(\mathbf{X}_0))^2\right)^{1/2} < \epsilon \text{ if } |\mathbf{X} - \mathbf{X}_0| < \delta \text{ and } \mathbf{X} \in D_{\mathbf{F}}.$$

Conversely, if this is true, then

$$|f_i(\mathbf{X}) - f_i(\mathbf{X}_0)| < \epsilon,\ 1 \le i \le m,\ \text{ if } |\mathbf{X} - \mathbf{X}_0| < \delta \text{ and } \mathbf{X} \in D_{\mathbf{F}}.$$

$$\lim_{\mathbf{X}\to\mathbf{X}_0} \frac{\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) - \mathbf{F}'(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0)}{|\mathbf{X} - \mathbf{X}_0|} = \mathbf{0}:$$

6.2.2. **(a)**

$$\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) - \mathbf{F}'(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) = \begin{bmatrix} 3x + 4y \\ 2x - y \\ x + y \end{bmatrix} - \begin{bmatrix} 3x_0 + 4y_0 \\ 2x_0 - y_0 \\ x_0 + y_0 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) $\mathbf{F}(\mathbf{X}) = \begin{bmatrix} 2x^2 + xy + 1 \\ xy \\ x^2 + y^2 \end{bmatrix}$; $\mathbf{F}(\mathbf{X}_0) = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 4x + y & x \\ y & x \\ 2x & 2y \end{bmatrix}$;

$$\mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix};$$

$$\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) - \mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0) = \begin{bmatrix} (x-1)(2x+y-1) \\ (x-1)(y+1) \\ (x-1)^2+(y+1)^2 \end{bmatrix}. \qquad \text{(A)}$$

Since $|(x-1)(y+1)| \le |\mathbf{X}-\mathbf{X}_0|^2$ and

$$|(x-1)(2x+y-1)| \le |\mathbf{X}-\mathbf{X}_0||2(x-1)+(y+1)| \le \sqrt{5}|\mathbf{X}-\mathbf{X}_0|^2,$$

(Schwarz's inequality), (A) implies that

$$|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) - \mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)| \le \sqrt{7}|\mathbf{X}-\mathbf{X}_0|^2,$$

which implies the conclusion.

(c) $\mathbf{F}(\mathbf{X}) = \begin{bmatrix} \sin(x+y) \\ \sin(y+z) \\ \sin(x+z) \end{bmatrix}$; $\mathbf{F}(\mathbf{X}_0) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$;

$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} \cos(x+y) & \cos(x+y) & 0 \\ 0 & \cos(y+z) & \cos(y+z) \\ \cos(x+z) & 0 & \cos(x+z) \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}_0) = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$;

$$\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) - \mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0) = \begin{bmatrix} \sin(x+y) - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x+y-\frac{\pi}{4}\right) \\ \sin(y+z) - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(y+z-\frac{\pi}{4}\right) \\ \sin(x+z-1) \end{bmatrix}. \quad \text{(A)}$$

From Taylor's theorem, $\left|\sin u - \frac{1}{\sqrt{2}}\left(u-\frac{\pi}{4}\right)\right| \le \frac{1}{2}\left(u-\frac{\pi}{4}\right)^2$, so

$$\left|\sin(x+y) - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x+y-\frac{\pi}{4}\right)\right| \le \frac{1}{2}\left[\left(x-\frac{\pi}{4}\right)+y\right]^2 \le |\mathbf{X}-\mathbf{X}_0|^2 \qquad \text{(B)}$$

and

$$\left|\sin(y+z) - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(y+z-\frac{\pi}{4}\right)\right| \le \frac{1}{2}\left[y+\left(z-\frac{\pi}{4}\right)\right]^2 \le |\mathbf{X}-\mathbf{X}_0|^2. \qquad \text{(C)}$$

From Taylor's theorem, $|\sin u - 1| \le \frac{1}{2}\left(u-\frac{\pi}{2}\right)^2$, so

$$|\sin(x+z)-1| \le \frac{1}{2}\left[\left(x-\frac{\pi}{4}\right)+\left(z-\frac{\pi}{4}\right)\right]^2 \le |\mathbf{X}-\mathbf{X}_0|^2. \qquad \text{(D)}$$

Now (A), (B), (C), and (D) imply that

$$\left|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) - \mathbf{F}'(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0)\right| \le \sqrt{3}|\mathbf{X} - \mathbf{X}_0|^2,$$

which implies the conclusion.

6.2.3. Since $\mathbf{F} = (f_1, f_2, \dots, f_m)$ is continuous at $\mathbf{X}_0$, $f_1, f_2, \dots, f_m$ are continuous at $\mathbf{X}_0$ (Definition 5.2.9). Since h is also continuous at $\mathbf{X}_0$, $hf_1, hf_2, \dots, hf_2$ are continuous at $\mathbf{X}_0$ (Theorem 5.2.8). Therefore, $h\mathbf{F} = (hf_1, hf_2, \dots, hf_m)$ is continuous at $\mathbf{X}_0$ (Definition 5.2.9).

6.2.4. Let $\mathbf{F} = (f_1, f_2, \dots, f_m)$ and $\mathbf{G} = (g_1, g_2, \dots, g_m)$. Since $\mathbf{F}$ and $\mathbf{G}$ are continuous at $\mathbf{X}_0$, $f_1, f_2, \dots, f_m$ and $g_1, g_2, \dots, g_m$ are continuous at $\mathbf{X}_0$ (Definition 5.2.9). Therefore, $f_1 + g_1, f_2 + g_2, \dots, g_1 + g_2$ are continuous at $\mathbf{X}_0$ (Theorem 5.2.8). Therefore, $\mathbf{F} + \mathbf{G} = (f_1 + g_1, f_2 + g_2, \dots, f_m + g_m)$ is continuous at $\mathbf{X}_0$ (Definition 5.2.9).

6.2.5. $\mathbf{H} = (h_1, h_2, \dots, h_m)$ where $h_i(\mathbf{U}) = f_i(g_1(\mathbf{U}), g_2(\mathbf{U}), \dots, g_n(\mathbf{U}))$, $1 \le i \le m$. From Theorem 5.2.11, h_i is continuous at $\mathbf{U}_0$. Therefore, $\mathbf{H}$ is continuous at $\mathbf{U}_0$ (Definition 5.2.9).

6.2.6. If $\mathbf{F} = (f_1, f_2, \dots, f_m)$ then $f_1, f_2, \dots, f_m$ are continuous on S (Definition 5.2.9). Hence, $h = f_1^2 + f_2^2 + \cdots + f_m^2$ is continuous on S (Theorem 5.2.8), so $|\mathbf{F}| = \sqrt{h}$ is continuous on S (Theorem 5.2.11).

6.2.7. Since $|\mathbf{F}|$ is continuous on S (Exercise 6.2.6), Theorem 5.2.13 implies the conclusion.

6.2.8. If $\mathbf{L}(\mathbf{X}) = \mathbf{A}\mathbf{X}$, then $|\mathbf{L}(\mathbf{X}) - \mathbf{L}(\mathbf{X}_0)| = |\mathbf{A}(\mathbf{X} - \mathbf{X}_0)| \le \|\mathbf{A}\||\mathbf{X} - \mathbf{X}_0|$.

6.2.9. **(a)** Since $\mathbf{L}$ is continuous on $\mathbb{R}^n$ (Exercise 6.2.8), there are vectors $\mathbf{Y}_0$ and $\mathbf{Y}_1$ in S such that $|\mathbf{A}\mathbf{Y}_0| \le |\mathbf{A}\mathbf{Y}| \le |\mathbf{A}\mathbf{Y}_1|$, $\mathbf{Y} \in S$ (Exercise 6.2.7); that is, $|\mathbf{A}\mathbf{Y}_0| = \min\left\{|\mathbf{A}\mathbf{Y}| \,\big|\, |\mathbf{Y}| = 1\right\}$ and $|\mathbf{A}\mathbf{Y}_1| = \max\left\{|\mathbf{A}\mathbf{Y}| \,\big|\, |\mathbf{Y}| = 1\right\}$. If $\mathbf{X} \ne \mathbf{0}$ then $\mathbf{Y} = \dfrac{\mathbf{X}}{|\mathbf{X}|} \in S$, so $|\mathbf{A}\mathbf{Y}_0| = \min\left\{\dfrac{|\mathbf{A}\mathbf{X}|}{|\mathbf{X}|} \,\Big|\, \mathbf{X} \ne \mathbf{0}\right\}$ and $|\mathbf{A}\mathbf{Y}_1| = \max\left\{\dfrac{|\mathbf{A}\mathbf{X}|}{|\mathbf{X}|} \,\Big|\, \mathbf{X} \ne \mathbf{0}\right\}$. This implies the conclusion, with $m(A) = |\mathbf{A}\mathbf{Y}_0|$ and $M(A) = |\mathbf{A}\mathbf{Y}_1|$.

(b) From **(a)**, $|\mathbf{A}\mathbf{X}| \le M(\mathbf{A})|\mathbf{X}|$ for all $\mathbf{X}$; therefore, (A) $\|\mathbf{A}\| \le M(\mathbf{A})$. Now suppose that $\epsilon > 0$. Since $\mathbf{L}$ is continuous on $\mathbb{R}^n$, there is a $\delta > 0$ such that $|\mathbf{A}\mathbf{Y}| > M(\mathbf{A}) - \epsilon$ if $\mathbf{Y} \in S$ and $|\mathbf{Y} - \mathbf{Y}_1| < \delta$. Therefore, $M(\mathbf{A}) - \epsilon < \|\mathbf{A}\|$. Letting $\epsilon \to 0+$ yields $M(\mathbf{A}) \le \|\mathbf{A}\|$. This and (A) imply that $M(\mathbf{A}) = \|\mathbf{A}\|$.

(c) If $n > m$ or $\mathbf{A}$ is singular the system $\mathbf{A}\mathbf{X} = \mathbf{0}$ has a nontrivial solution, which we may assume without loss of generality to be in S. Therefore, $m(\mathbf{A}) = 0$.

(d) Applying **(a)** to $\mathbf{A}^{-1}$ shows that there is $\mathbf{U}_1 \in S$ such that $|\mathbf{A}^{-1}\mathbf{U}_1| = M(\mathbf{A}^{-1})$. Since $\mathbf{V}_1 = \dfrac{\mathbf{A}^{-1}\mathbf{U}_1}{M(\mathbf{A}^{-1})} \in S$, $m(\mathbf{A}) \le |\mathbf{A}\mathbf{V}_1| = \dfrac{|\mathbf{U}_1|}{M(\mathbf{A}^{-1})} = \dfrac{1}{M(\mathbf{A}^{-1})}$. Therefore, (A) $m(\mathbf{A})M(\mathbf{A}^{-1}) \le 1$. From **(a)**, there is a $\mathbf{U}_2 \in S$ such that $|\mathbf{A}\mathbf{U}_2| = m(\mathbf{A})$. Since $\mathbf{V}_2 = \dfrac{\mathbf{A}\mathbf{U}_2}{m(\mathbf{A})} \in S$, $M(\mathbf{A}^{-1}) \ge \dfrac{|\mathbf{U}_2|}{m(\mathbf{A})} = \dfrac{1}{m(\mathbf{A})}$. Therefore, $m(\mathbf{A})M(\mathbf{A}^{-1}) \ge 1$. This and (A) imply that $m(\mathbf{A})M(\mathbf{A}^{-1}) = 1$. Applying this result to $\mathbf{A}^{-1}$ yields $m(\mathbf{A}^{-1})M(\mathbf{A}) = 1$.

6.2.10. Let $\mathbf{F} = (f_1, f_2, \dots, f_m)$ and $\epsilon > 0$. Since $f_1, f_2, \dots, f_m$ are uniformly continuous on S, there is a $\delta > 0$ such that

$$|f_i(\mathbf{X}) - f_i(\mathbf{Y})| \le \frac{\epsilon}{\sqrt{m}}, \quad 1 \le i \le m, \quad \text{if} \quad |\mathbf{Y} - \mathbf{X}| < \delta \quad \text{and} \quad \mathbf{X}, \mathbf{Y} \in S.$$

Then $|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{Y})| < \epsilon$ if $|\mathbf{X} - \mathbf{Y}| < \delta$ and $\mathbf{X}, \mathbf{Y} \in S$.

6.2.11. Must show that if a is an arbitrary real number, then (A) $\mathbf{F}(a\mathbf{X}) = a\mathbf{F}(\mathbf{X})$. First consider $a = 0$: $\mathbf{F}(0\mathbf{X}) = \mathbf{F}(\mathbf{0}) = \mathbf{F}(\mathbf{0} + \mathbf{0}) = \mathbf{F}(\mathbf{0}) + \mathbf{F}(\mathbf{0}) = 2\mathbf{F}(\mathbf{0}) = 2\mathbf{F}(0\mathbf{X})$; hence, $\mathbf{F}(0\mathbf{X}) = 2\mathbf{F}(0\mathbf{X})$, which implies (A) if $a = 0$. Since $F[(n+1)\mathbf{X}] = \mathbf{F}(n\mathbf{X}) + \mathbf{F}(\mathbf{X})$, induction yields (A) if a is a nonnegative integer. If n is a negative integer, then

$$\begin{aligned} 0 = \mathbf{F}(0\mathbf{X}) &= \mathbf{F}(n\mathbf{X} + |n|\mathbf{X}) = \mathbf{F}(n\mathbf{X}) + \mathbf{F}(|n|\mathbf{X}) \\ &= \mathbf{F}(n\mathbf{X}) + |n|\mathbf{F}(\mathbf{X}) = \mathbf{F}(n\mathbf{X}) - n\mathbf{F}(\mathbf{X}); \end{aligned}$$

hence, (A) holds if a is any integer. If $a = m/n$, then

$$F\left(\frac{m}{n}\mathbf{X}\right) = mF\left(\frac{\mathbf{X}}{n}\right) = \frac{m}{n}F\left[n\left(\frac{\mathbf{X}}{n}\right)\right] = \frac{m}{n}\mathbf{F}(\mathbf{X});$$

hence, (A) holds if a is rational. If a is any real and r is rational, then $|\mathbf{F}(a\mathbf{X}) - a\mathbf{F}(\mathbf{X})| \le |\mathbf{F}(a\mathbf{X}) - \mathbf{F}(r\mathbf{X})| + |r - a||\mathbf{F}(\mathbf{X})|$. Let $r \to a$ and use the continuity of $\mathbf{F}$ to complete the proof.

6.2.15. Let $\mathbf{G}_i(\mathbf{X}) = \mathbf{A}_i + \mathbf{B}_i(\mathbf{X} - \mathbf{X}_0)$ $(i = 1, 2)$, and $\mathbf{X} = \mathbf{X}_0 + t\mathbf{U}$, with $|\mathbf{U}| = 1$. Then

$$\lim_{t \to 0} \frac{(\mathbf{A}_1 - \mathbf{A}_2) + t(\mathbf{B}_1 - \mathbf{B}_2)\mathbf{U}}{t} = 0,$$

which implies that $\mathbf{A}_1 = \mathbf{A}_2$ and $(\mathbf{B}_1 - \mathbf{B}_2)\mathbf{U} = 0$ for every $\mathbf{U}$. The latter implies that $\mathbf{B}_1 = \mathbf{B}_2$.

6.2.16. If $\mathbf{F}$ is differentiable at $\mathbf{X}_0$ then

$$\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) = \mathbf{F}'(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) + |\mathbf{X} - \mathbf{X}_0|\mathbf{E}(\mathbf{X}),$$

where $\lim_{\mathbf{X} \to \mathbf{X}_0} \mathbf{E}(\mathbf{X}) = \mathbf{0}$. Choose δ_1 such that $|\mathbf{E}(\mathbf{X})| < 1$ if $|\mathbf{X} - \mathbf{X}_0| < \delta_1$ and $\mathbf{X} \in D_{\mathbf{F}}$. Then $|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0)| < (\|\mathbf{F}'(\mathbf{X}_0)\| + 1)|\mathbf{X} - \mathbf{X}_0|$ if $|\mathbf{X} - \mathbf{X}_0| < \delta_1$. Therefore, $\mathbf{F}$ is continuous at $\mathbf{X}_0$

6.2.17. If $\mathbf{F}$ is differentiable at $\mathbf{X}_0$ then

$$\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) = \mathbf{F}'(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) + |\mathbf{X} - \mathbf{X}_0|\mathbf{E}(\mathbf{X}), \quad \mathbf{X} \in D_{\mathbf{F}},$$

where $\lim_{\mathbf{X} \to \mathbf{X}_0} \mathbf{E}(\mathbf{X}) = \mathbf{0}$. If $\epsilon > 0$ choose δ so that $\mathbf{X} \in D_{\mathbf{F}}$ and $|\mathbf{E}(\mathbf{X})| < \epsilon$ if $|\mathbf{X} - \mathbf{X}_0| < \delta$. Then

$$|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0)| \le (\|\mathbf{F}'(\mathbf{X}_0)\| + \epsilon)|\mathbf{X} - \mathbf{X}_0| \quad \text{if} \quad |\mathbf{X} - \mathbf{X}_0| < \delta.$$

6.2.18. If $\mathbf{F}$ is differentiable at $\mathbf{X}_0$ then

$$\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) = \mathbf{F}'(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) + |\mathbf{X} - \mathbf{X}_0|\mathbf{E}(\mathbf{X}), \quad \mathbf{X} \in D_{\mathbf{F}}, \tag{A}$$

where $\lim_{\mathbf{X}\to\mathbf{X}_0} \mathbf{E}(\mathbf{X}) = \mathbf{0}$. Since

$$|\mathbf{X}-\mathbf{X}_0| = |[\mathbf{F}'(\mathbf{X}_0)]^{-1}\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)| \le \frac{1}{r}|\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)|,$$

$|\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)| \ge r|\mathbf{X}-\mathbf{X}_0|$. If $0<\epsilon<r$ choose δ so that $\mathbf{X}\in D_{\mathbf{F}}$ and $|\mathbf{E}(\mathbf{X})|<\epsilon$ if $|\mathbf{X}-\mathbf{X}_0|<\delta$. Now (A) implies that

$$|\mathbf{F}(\mathbf{X})-\mathbf{F}(\mathbf{X}_0)| \ge (r-\epsilon)|\mathbf{X}-\mathbf{X}_0| \quad \text{if} \quad |\mathbf{X}-\mathbf{X}_0|<\delta.$$

6.2.19. $|\mathbf{X}-\mathbf{Y}| = |\mathbf{A}^{-1}(\mathbf{L}(\mathbf{X})-\mathbf{L}(\mathbf{Y}))| \le \|\mathbf{A}^{-1}\||\mathbf{L}(\mathbf{X})-\mathbf{L}(\mathbf{Y})|$.

6.2.20.**(a)** $\mathbf{G}'(\mathbf{U}) = \begin{bmatrix} -w\sin u\sin v & w\cos u\cos v & \cos u\sin v \\ w\cos u\sin v & w\sin u\cos v & \sin u\sin v \\ 0 & -w\sin v & \cos v \end{bmatrix}$; $\mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}$;

$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 2x & 2y & 0 \\ -\dfrac{2xz}{(x^2+y^2)^2} & -\dfrac{2yz}{(x^2+y^2)^2} & \dfrac{1}{x^2+y^2} \end{bmatrix}$; $\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$;

$\mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$;

$\mathbf{H}'(\mathbf{U}_0) = \mathbf{F}'(\mathbf{X}_0)\mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$.

Check:

$\mathbf{H}(\mathbf{U}) = \begin{bmatrix} w^2\sin^2 v \\ \dfrac{\cos v}{w\sin^2 v} \end{bmatrix}$; $\mathbf{H}'(\mathbf{U}) = \begin{bmatrix} 0 & 2w^2\sin v\cos v & 2w\sin^2 v \\ 0 & -\dfrac{1+\cos^2 v}{w\sin^3 v} & -\dfrac{\cos v}{w^2\sin^2 v} \end{bmatrix}$; $\mathbf{H}'(\mathbf{U}_0) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$.

(b) $\mathbf{G}'(\mathbf{U}) = \begin{bmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \end{bmatrix}$; $\mathbf{G}'(\mathbf{U}_0) = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -3 & 1 \\ 3 & 1 \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 2x & -2y \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{bmatrix}$;

$\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0) = \dfrac{3}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} \frac{6}{\sqrt{2}} & -\frac{6}{\sqrt{2}} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \end{bmatrix}$;

$\mathbf{H}'(\mathbf{U}_0) = \mathbf{F}'(\mathbf{X}_0)\mathbf{G}'(\mathbf{U}_0) = \dfrac{1}{\sqrt{2}}\begin{bmatrix} \frac{6}{\sqrt{2}} & -\frac{6}{\sqrt{2}} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -18 & 0 \\ 2 & 0 \end{bmatrix}$.

Check:

$\mathbf{H}(\mathbf{U}) = \begin{bmatrix} v^2\cos 2u \\ \tan u \end{bmatrix}$; $\mathbf{H}'(\mathbf{U}) = \begin{bmatrix} -2v^2\sin 2u & 2v\cos 2u \\ \sec^2 u & 0 \end{bmatrix}$; $\mathbf{H}'(\mathbf{U}_0) = \begin{bmatrix} -18 & 0 \\ 2 & 0 \end{bmatrix}$.

(c) $\mathbf{G}'(\mathbf{U}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 3 & 4 & 2 \\ 4 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$;

$$\mathbf{H}'(\mathbf{U}) = \mathbf{F}'(\mathbf{X})\mathbf{G}'(\mathbf{U}) = \begin{bmatrix} 3 & 4 & 2 \\ 4 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ 3 & -8 \\ 1 & 0 \end{bmatrix}.$$

Check:

$$\mathbf{H}(\mathbf{U}) = \begin{bmatrix} 9u - 3v + 6 \\ 3u - 8v - 1 \\ u - 2 \end{bmatrix}; \mathbf{H}'(\mathbf{U}) = \begin{bmatrix} 9 & -3 \\ 3 & -8 \\ 1 & 0 \end{bmatrix}.$$

(d) $\mathbf{G}'(\mathbf{U}) = \begin{bmatrix} 2 & -1 & 1 \\ 2ue^{u^2-v^2} & -2ve^{u^2-v^2} & 0 \end{bmatrix}; \mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} 2 & -1 & 1 \\ 2 & -2 & 0 \end{bmatrix};$

$$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$\mathbf{H}'(\mathbf{U}_0) = \mathbf{F}'(\mathbf{X}_0)\mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check:

$$\mathbf{H}(\mathbf{U}) = \begin{bmatrix} 2u - v + w + e^{u^2-v^2} \\ 2u - v + w - e^{u^2-v^2} \end{bmatrix}; \mathbf{H}'(\mathbf{U}_0) = \begin{bmatrix} 4 & -3 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(e) $\mathbf{G}'(\mathbf{U}) = \begin{bmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{bmatrix}; \mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix};$

$$\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix};$$

$$\mathbf{H}'(\mathbf{U}_0) = \mathbf{F}'(\mathbf{X}_0)\mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}.$$

Check:

$$\mathbf{H}(\mathbf{U}) = \begin{bmatrix} e^{2u} \\ e^{2u} \cos 2v \end{bmatrix}; \mathbf{H}'(\mathbf{U}) = \begin{bmatrix} 2e^{2u} & 0 \\ 2e^{2u} \cos 2v & -2e^{2u} \sin 2v \end{bmatrix};$$

$$\mathbf{H}'(\mathbf{U}_0) = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}.$$

(f) $\mathbf{G}'(\mathbf{U}) = \begin{bmatrix} 1 & 2 \\ 2 & -2v \end{bmatrix}; \mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}; \mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 1 & 2 \\ 1 & -2y \\ 2x & 1 \end{bmatrix};$

$$\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0) = \begin{bmatrix} -3 \\ -2 \end{bmatrix}; \mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ -6 & 1 \end{bmatrix};$$

$$\mathbf{H}'(\mathbf{U}_0) = \mathbf{F}'(\mathbf{X}_0)\mathbf{G}'(\mathbf{U}_0) = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 9 & 18 \\ -4 & -8 \end{bmatrix}.$$

Check:

$$\mathbf{H}(\mathbf{U}) = \begin{bmatrix} 5u + 2v - 2v^2 \\ -4u^2 + 4uv^2 + u - v^4 + 2v \\ u^2 + 4uv + 2u + 3v^2 \end{bmatrix};$$

$$\mathbf{H}'(\mathbf{U}) = \begin{bmatrix} 5 & 2-4v \\ -8u+4v^2+1 & 8uv-4v^3+2 \\ 2u+4v+2 & 4u+6v \end{bmatrix}; \mathbf{H}'(\mathbf{U}_0) = \begin{bmatrix} 5 & 10 \\ 9 & 18 \\ -4 & -8 \end{bmatrix}.$$

6.2.21. From Theorem 6.2.8, $\mathbf{H}'(\mathbf{U}) = \mathbf{F}'(\mathbf{G}(\mathbf{U}))\mathbf{G}'(\mathbf{U})$. Since the determinant of a product of two matrices is the product of their determinants,

$$\frac{\partial(h_1,h_2,\dots,h_n)}{\partial(u_1,u_2,\dots,u_n)} = \frac{\partial(f_1,f_2,\dots,f_n)}{\partial(x_1,x_2,\dots,x_n)}\frac{\partial(g_1,g_2,\dots,g_n)}{\partial(u_1,u_2,\dots,u_n)}.$$

where $\dfrac{\partial(g_1,g_2,\dots,g_n)}{\partial(u_1,u_2,\dots,u_n)}$ and $\dfrac{\partial(h_1,h_2,\dots,h_n)}{\partial(u_1,u_2,\dots,u_n)}$ are evalutaed at $\mathbf{U}$ and $\dfrac{\partial(f_1,f_2,\dots,f_n)}{\partial(x_1,x_2,\dots,x_n)}$ is evaluated at $\mathbf{G}(\mathbf{U})$.

6.2.22. If $\mathbf{F} = (f_1, f_2, \dots, f_m)$ then $\mathbf{F}$ is continuous at $\mathbf{X}_0$ if and only if $f_1, f_2, \dots, f_m$ are all continuous at $\mathbf{X}_0$. Now apply Exercise 5.2.15.

6.2.23. $\mathbf{F}(S)$ is bounded, by Exercise 6.2.7. Hence, we need only show that $\mathbf{F}(S)$ is closed. If $\overline{\mathbf{Y}}$ is a limit point of $\mathbf{F}(S)$, there is a sequence $\{\mathbf{Y}_k\}$ in $\mathbf{F}(S)$ such that $\lim_{k\to\infty}\mathbf{Y}_k = \overline{\mathbf{Y}}$, and a sequence $\{\mathbf{X}_k\}$ in S such that $\mathbf{F}(\mathbf{X}_k) = \mathbf{Y}_k$. Since S is compact, $\{\mathbf{X}_k\}$ has a subsequence $\{\mathbf{X}_{k_j}\}$ which converges to a point $\overline{\mathbf{X}}$ in S. From the continuity of $\mathbf{F}$, $\mathbf{F}(\overline{\mathbf{X}}) = \overline{\mathbf{Y}}$. Therefore, $\overline{\mathbf{Y}} \in \mathbf{F}(S)$. Since $\mathbf{F}(S)$ contains all its limit points, $\mathbf{F}(S)$ is closed.

6.3 THE INVERSE FUNCTION THEOREM

6.3.1. If $\mathbf{F}(\mathbf{G}_1(\mathbf{U})) = \mathbf{F}(\mathbf{G}_2(\mathbf{U})) = Y$ for all $\mathbf{U}$ in $R(F)$, then $\mathbf{G}_1(\mathbf{U}) = \mathbf{G}_2(\mathbf{U})$ for all such $\mathbf{U}$, since $\mathbf{F}$ is one-to-one.

6.3.2. Suppose that $\mathbf{L}$ is invertible and $\mathbf{AX}_0 = \mathbf{0}$. Since $\mathbf{L}(\mathbf{0}) = \mathbf{A0} = \mathbf{0}$ and $\mathbf{L}$ is one-to-one, $\mathbf{X}_0 = \mathbf{0}$. Therefore, $\mathbf{A}$ is nonsingular, by Theorem 6.1.15.

Conversely, suppose that $\mathbf{A}$ is nonsingular and $\mathbf{L}(\mathbf{X}_1) = \mathbf{L}(\mathbf{X}_2)$. Then $\mathbf{AX}_1 = \mathbf{AX}_2$, so $\mathbf{A}(\mathbf{X}_1 - \mathbf{X}_2) = \mathbf{0}$ and $\mathbf{X}_1 - \mathbf{X}_2 = \mathbf{0}$, by Theorem 6.1.15. Therefore, $\mathbf{X}_1 = \mathbf{X}_2$, so $\mathbf{L}$ is invertible.

If $\mathbf{A}$ is nonsingular and $\mathbf{U} \in \mathbb{R}^n$ then $\mathbf{L}(\mathbf{A}^{-1}\mathbf{U}) = \mathbf{U}$. This shows that $\mathbf{L}(\mathbb{R}^n) = \mathbb{R}^n$.

6.3.3. By Theorem 6.1.8, there is a nonzero vector $\mathbf{X}_0$ such that $\mathbf{AX}_0 = \mathbf{0}$. If S is open and $\overline{\mathbf{X}} \in S$, then $\overline{\mathbf{X}} + t\mathbf{X}_0 \in S$ for small t; since $\mathbf{A}(\overline{\mathbf{X}} + t\mathbf{X}_0) = \mathbf{A}(\overline{\mathbf{X}})$, $\mathbf{L}$ is not one-to-one on S.

6.3.6. **(a)** Suppose that $a < x_1 < x_2 < b$ and $f(x_1) = f(x_2) = L$. If f is nonconstant on $[x_1, x_2]$, f must attain at least one of its extreme values on $[x_1, x_2]$ at a point x_0 in (x_1, x_2). Suppose that $f(x_0) = M$ is a local maximum of f. We will show that f cannot be one-to-one on any neighborhood of x_0. If $\delta > 0$ and any two of $f(x_0 - \delta)$, $f(x_0 + \delta)$, and $f(x_0)$ are equal, then f is not one-to-one on $[x_0 - \delta, x_0 + \delta]$. If they are all different, suppose that (for example) that $f(x_0 + \delta) < f(x_0 - \delta) < f(x_0)$. Then applying the intermediate value theorem (Theorem 2.2.10) to $[x_0, x_0 + \delta]$ implies that $f(\overline{x}) = f(x_0 - \delta)$ for some $\overline{x}$ in this interval; thus f is not one-to-one on $[x_0 - \delta, x_0 + \delta]$.

6.3.7. Since $ax + by = \sqrt{a^2+b^2}\sqrt{x^2+y^2}\cos\theta$, where θ is the angle between the vectors (a, b) and (x, y), $(x, y) \neq (0, 0)$ is in S if and only if (x, y) has an argument satisfying

$$\phi - \pi/2 < \arg(x, y) < \phi + \pi/2,$$

where $\phi = \arg(a, b)$. Since $\mathbf{F}$ doubles arguments (see Example 6.3.4),

$$\mathbf{F}(S) = \left\{(u, v) \,\middle|\, 2\phi - \pi < \arg(u, v) < 2\phi + \pi\right\},$$

and $\mathbf{F}_S^{-1}$ is given by Eqn. (6.3.36), with $2\phi - \pi < \arg(u, v) < 2\phi + \pi$. Similarly, $(x, y) \neq (0, 0)$ is in S_1 if and only if x, y has an argument satisfying the inequalities

$$\phi + \pi/2 < \arg(x, y) < \phi + 3\pi/2,$$

and

$$\mathbf{F}(S_1) = \left\{(u, v) \,\middle|\, 2\phi + \pi < \arg(u, v) < 2\phi + 3\pi\right\}.$$

Clearly $\mathbf{F}(S_1) = \mathbf{F}(S)$, and $\mathbf{F}_{S_1}^{-1}$ is given by Eqn. (6.3.36), with $2\phi + \pi < \arg(u, v) < 2\phi + 3\pi$. Since the values of $\arg(u, v)$ used in $\mathbf{F}_S^{-1}$ and $\mathbf{F}_{S_1}^{-1}$ differ by 2π and the inverse mappings halve arguments, $\mathbf{F}_S^{-1} = -\mathbf{F}_{S_1}^{-1}$.

6.3.8. If $\mathbf{F}(x_1, u_1) = \mathbf{F}(x_2, u_2)$, then $e^{x_1} = |\mathbf{F}(x_1, u_1)| = |\mathbf{F}(x_2, u_2)| = e^{x_2}$, so $x_1 = x_2$. Hence, $\sin u_1 = \sin u_2$ and $\cos u_1 = \cos u_2$, which implies that $\sin(u_1 - u_2) = \cos(u_1 - u_2) = 0$; hence, $u_1 - u_2 = 2k\pi$ (k = integer). Because of the assumption on S, $k = 0$; that is, $u_1 = u_2$.

6.3.9. Since S is compact, Exercise 5.1.32 implies that a subsequence $\{\mathbf{F}_S^{-1}(\mathbf{U}_{k_j})\}$ converges to a limit $\mathbf{X}_0$ in S. From the construction of the sequence, $\mathbf{X}_0 \neq \mathbf{F}_S^{-1}(\overline{\mathbf{U}})$; however, $\mathbf{F}(\mathbf{X}_0) = \overline{\mathbf{U}}$ because $\lim_{j\to\infty} \mathbf{U}_{k_j} = \overline{\mathbf{U}}$ and $\mathbf{F}$ is continuous at $\mathbf{X}_0$. Since $\mathbf{F}\left(\mathbf{F}_S^{-1}(\overline{\mathbf{U}})\right) = \overline{\mathbf{U}}$ also, this contradicts the assumption that $\mathbf{F}$ is one-to one on S.

6.3.10. **(a)** $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$;

$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{F}^{-1}(u, v) = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \dfrac{1}{10} \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \dfrac{1}{10} \begin{bmatrix} u - 2v \\ 3u + 4v \end{bmatrix}$;

$(\mathbf{F}^{-1})' = \dfrac{1}{10} \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$;

(b) $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 1 & -4 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$;

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{F}^{-1}(u, v, w) = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 1 & -4 \\ -1 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \dfrac{1}{2} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} =$
$\dfrac{1}{2} \begin{bmatrix} u + 2v + 3w \\ u - w \\ u + v + 2w \end{bmatrix}$;

$(\mathbf{F}^{-1})' = \dfrac{1}{2} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$.

6.3.11. If $\mathbf{F} = (f_1, f_2, \dots, f_n)$ and $\mathbf{F}_S^{-1} = (g_1, \dots, g_n)$, then $\dfrac{\partial g_i}{\partial u_j}(\mathbf{U})$ can be written as the quotient of determinants with entries of the form $\dfrac{\partial f_r}{\partial x_s}\left(\mathbf{F}_S^{-1}(\mathbf{U})\right)$. Repeated use of the chain rule gives the result.

6.3.12. $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{F}(x, y) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$;

$\mathbf{F}'(x, y) = 2\begin{bmatrix} x & y \\ x & -y \end{bmatrix}$; if $xy \neq 0$, then $(F'(x, y))^{-1} = \dfrac{1}{4}\begin{bmatrix} 1/x & 1/x \\ 1/y & -1/y \end{bmatrix}$;

$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} u+v \\ u-v \end{bmatrix}$; therefore, the four branches of $\mathbf{F}^{-1}$ are $\mathbf{G}_1(u, v) = \dfrac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{u+v} \\ \sqrt{u-v} \end{bmatrix}$, $\mathbf{G}_2(u, v) = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -\sqrt{u+v} \\ \sqrt{u-v} \end{bmatrix}$, $\mathbf{G}_3(u, v) = \dfrac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{u+v} \\ -\sqrt{u-v} \end{bmatrix}$, $\mathbf{G}_4(u, v) = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -\sqrt{u+v} \\ -\sqrt{u-v} \end{bmatrix}$, with differential matrices

$$\mathbf{G}_1'(u, v) = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1/\sqrt{u+v} & 1/\sqrt{u+v} \\ 1/\sqrt{u-v} & -1/\sqrt{u-v} \end{bmatrix} = (\mathbf{F}'(x(u, v), y(u, v)))^{-1},$$

$$\mathbf{G}_2'(u, v) = \frac{1}{2\sqrt{2}}\begin{bmatrix} -1/\sqrt{u+v} & -1/\sqrt{u+v} \\ 1/\sqrt{u-v} & -1/\sqrt{u-v} \end{bmatrix} = (\mathbf{F}'(x(u, v), y(u, v)))^{-1},$$

$$\mathbf{G}_3'(u, v) = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1/\sqrt{u+v} & 1/\sqrt{u+v} \\ -1/\sqrt{u-v} & 1/\sqrt{u-v} \end{bmatrix} = (\mathbf{F}'(x(u, v), y(u, v)))^{-1},$$

$$\mathbf{G}_4'(u, v) = \frac{1}{2\sqrt{2}}\begin{bmatrix} -1/\sqrt{u+v} & -1/\sqrt{u+v} \\ -1/\sqrt{u-v} & 1/\sqrt{u-v} \end{bmatrix} = (\mathbf{F}'(x(u, v), y(u, v)))^{-1}.$$

6.3.13. **(a)** To see that $\mathbf{F}$ is one-to-one on S suppose that $\mathbf{X}_1 = (x_{11}, x_{21}, \dots, x_{n1})$ and $\mathbf{X}_2 = (x_{12}, x_{22}, \dots, x_{n2})$ are in S and $\mathbf{F}(\mathbf{X}_1) = \mathbf{F}(\mathbf{X}_2)$. Then, since $\mathbf{A}$ is invertible, $x_{i2}^2 = x_{i1}^2$, or, equivalently, (A) $(x_{i2} - x_{i1})(x_{i2} + x_{i1}) = 0$, $1 \le i \le n$. Since $e_i x_{i1} > 0$ and $e_i x_{i2} > 0$, $x_{i1} + x_{i2} \neq 0$. Therefore, (A) implies that $x_{i1} = x_{i2}$, so $\mathbf{X}_1 = \mathbf{X}_2$.

Now we note that

$$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n \end{bmatrix}$$

is nonsingular on S, since the determinant of this matrix is $x_1 x_2 \cdots x_n \det(A) \neq 0$.

(b) Let $\mathbf{A}^{-1} = [b_{ij}]$. Since

$$\begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{bmatrix} = \mathbf{BU},$$

$R(\mathbf{F}) = \left\{\mathbf{U} \,\middle|\, \sum_{j=1}^n b_{ij}u_j > 0,\ 1 \le i \le n\right\}$, and $\mathbf{X} = \mathbf{F}_S^{-1}(\mathbf{U})$ is defined by (A) $x_i =$

$e_i\left(\sum_{j=1}^n b_{ij}u_j\right)^{1/2}$, $1 \le i \le n$;

$$(\mathbf{F}_S)'(\mathbf{U}) = (\mathbf{F}'(\mathbf{X}))^{-1} = \frac{1}{2}\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n \end{bmatrix}^{-1},$$

with $x_1, x_2, \dots, x_n$ as in (A).

6.3.14. **(a)** Since the origin is not in S, Eqn. (6.3.35) implies that we cannot have $\theta_x(x,y) = \theta_y(x,y) = 0$ for any (x,y) in S.

(b) Since $a > 0$ (because the origin is not in S), the possible arguments of any two points on the line segment can differ only by multiples of 2π. Since θ is continuous on the segment, θ must be constant.

(c) Suppose that $A \subset S$. Since A is compact and θ is continuous on A, θ assumes a maximum on A. From **(b)**, θ must assume this maximum in A^0, which contradicts **(a)**.

(d) Every deleted neighborhood of $(0,0)$ contains a subset like A in **(c)**.

6.3.16. Since $au+bv = \sqrt{a^2+b^2}\sqrt{u^2+v^2}\cos\phi$ where is the angle between the vectors (a,b) and (u,v), $(u,v) \ne (0,0)$ is in T if and only if (u,v) has an argument that satisfies $\beta - \pi/2 < \arg(u,v) < \beta + \pi/2$. Therefore, from Example 6.3.8,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{G}(u,v) = (u^2+v^2)^{1/4}\begin{bmatrix} \cos[\frac{1}{2}\arg(u,v)] \\ \sin(\arg(u,v)/2) \end{bmatrix},$$

where $\beta - \pi/2 < \arg(u,v) < \beta + \pi/2$. From the argument in Example 6.3.8,

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} = \frac{x}{2(x^2+y^2)} = \frac{1}{2}(u^2+v^2)^{-1/4}\cos(\arg(u,v)/2)$$

and

$$\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u} = \frac{y}{2(x^2+y^2)} = \frac{1}{2}(u^2+v^2)^{-1/4}\sin(\arg(u,v)/2).$$

To obtain these formulas directly, we note that

$$\frac{u}{(u^2+v^2)^{1/2}} = \cos(\arg(u,v)) \quad \text{and} \quad \frac{v}{(u^2+v^2)^{1/2}} = \sin(\arg(u,v)), \qquad \text{(A)}$$

while

$$\frac{\partial \arg(u,v)}{\partial u} = -\frac{v}{u^2+v^2} \quad \text{and} \quad \frac{\partial \arg(u,v)}{\partial v} = \frac{u}{u^2+v^2} \qquad \text{(B)}$$

(from (35) with (x,y) replaced by (u,v)).

Since $x = (u^2 + v^2)^{1/4} \cos(\arg(u, v)/2)$,

$$\begin{aligned}\frac{\partial x}{\partial u} &= \frac{u}{2(u^2+v^2)^{3/4}} \cos(\arg(u,v)/2) - (u^2+v^2)^{1/4} \sin(\arg(u,v)/2) \left(-\frac{v}{2(u^2+v^2)}\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\frac{u}{(u^2+v^2)^{1/2}} \cos(\arg(u,v)/2) + \frac{v}{(u^2+v^2)^{1/2}} \sin(\arg(u,v)/2)\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\cos(\arg(u,v)) \cos(\arg(u,v)/2) + \sin(\arg(u,v)) \sin(\arg(u,v)/2)\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \cos(\arg(u,v)/2),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial x}{\partial v} &= \frac{v}{2(u^2+v^2)^{3/4}} \cos(\arg(u,v)/2) - (u^2+v^2)^{1/4} \sin(\arg(u,v)/2) \left(\frac{u}{2(u^2+v^2)}\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\frac{v}{(u^2+v^2)^{1/2}} \cos(\arg(u,v)/2) - \frac{u}{(u^2+v^2)^{1/2}} \sin(\arg(u,v)/2)\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\sin(\arg(u,v)) \cos(\arg(u,v)/2) - \cos(\arg(u,v)) \sin(\arg(u,v)/2)\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \sin(\arg(u,v)/2).\end{aligned}$$

Since $y = (u^2 + v^2)^{1/4} \sin(\arg(u, v)/2)$,

$$\begin{aligned}\frac{\partial y}{\partial u} &= \frac{u}{2(u^2+v^2)^{3/4}} \sin(\arg(u,v)/2) + (u^2+v^2)^{1/4} \cos(\arg(u,v)/2) \left(-\frac{v}{2(u^2+v^2)}\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\frac{u}{(u^2+v^2)^{1/2}} \sin(\arg(u,v)/2) - \frac{v}{(u^2+v^2)^{1/2}} \cos(\arg(u,v)/2)\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\cos(\arg(u,v)) \sin(\arg(u,v)/2) - \sin(\arg(u,v)) \cos(\arg(u,v)/2)\right) \\ &= -\frac{1}{2(u^2+v^2)^{1/4}} \sin(\arg(u,v)/2),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial y}{\partial v} &= \frac{v}{2(u^2+v^2)^{3/4}} \sin(\arg(u,v)/2) + (u^2+v^2)^{1/4} \cos(\arg(u,v)/2) \left(\frac{u}{2(u^2+v^2)}\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\frac{v}{(u^2+v^2)^{1/2}} \sin(\arg(u,v)/2) + \frac{u}{(u^2+v^2)^{1/2}} \cos(\arg(u,v)/2)\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \left(\sin(\arg(u,v)) \sin(\arg(u,v)/2) + \cos(\arg(u,v)) \cos(\arg(u,v)/2)\right) \\ &= \frac{1}{2(u^2+v^2)^{1/4}} \cos(\arg(u,v)/2).\end{aligned}$$

6.3.17. $\mathbf{F}_S^{-1}$ is regular on $\mathbf{F}(S)$, by Theorem 6.3.3. If $\mathbf{F}_S^{-1}(u,v) = (x(u,v), y(u,v))$, then

$$\begin{bmatrix} x_u & x_v \\ u_u & u_v \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}^{-1} = \frac{1}{u_x v_y - v_x u_y} \begin{bmatrix} v_y & -u_y \\ -v_x & u_x \end{bmatrix},$$

which yields the conclusion.

6.3.18. Since $\mathbf{G}(\mathbf{F}(\mathbf{X})) = \mathbf{X}$, $\mathbf{G}'(\mathbf{F}(\mathbf{X}))\mathbf{F}'(\mathbf{X}) = \mathbf{I}$ (Theorem 6.2.8). Since $\det(\mathbf{G}') = \dfrac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)}$, $\det(\mathbf{F}') = \dfrac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)}$, and $\det(\mathbf{I}) = 1$, Theorem 6.1.9 implies the conclusion, with $\dfrac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)}$ evaluated at $\mathbf{X}$ and $\dfrac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)}$ evaluated at $\mathbf{U} = \mathbf{F}(\mathbf{X})$ or, equivalently, with $\dfrac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)}$ evaluated at $\mathbf{U}$ and $\dfrac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)}$ evaluated at $\mathbf{X} = \mathbf{G}(\mathbf{U})$.

6.3.19. If $\mathbf{F}(\mathbf{X}) = (x_1^3, x_2^3, \dots, x_n^3)$ then

$$J\mathbf{F} = 3 \begin{bmatrix} x_1^2 & 0 & 0 \cdots & 0 & \\ 0 & x_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n^2 \end{bmatrix}$$

so $J\mathbf{F}(\mathbf{0}) = 0$. However, since the function $y = x^3$ is one-to-one on $(-\infty, \infty)$, $\mathbf{F}$ is one-to-one on $\mathbb{R}^n$.

More generally, let $\mathbf{F} = (f_1, f_2, \dots, f_n)$ where f_1, f_2, $\dots f_n$ are all strictly monotonic differentiable functions on $(-\infty, \infty)$ and f_1', f_2', $\dots$, f_n' have one or more common zeros.

6.3.20. In all cases, $\mathbf{A}(\mathbf{U}) = \mathbf{G}(\mathbf{U}_0) + \mathbf{G}'(\mathbf{U}_0)(\mathbf{U} - \mathbf{U}_0) = \mathbf{X}_0 + (\mathbf{F}'(\mathbf{X}_0))^{-1}(\mathbf{U} - \mathbf{U}_0)$.

(a) $\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 4x^3y^5 - 4 & 5x^4y^4 \\ 3x^2y^2 & 2x^3y - 3 \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} -8 & 5 \\ 3 & -5 \end{bmatrix}$;

$(\mathbf{F}'(\mathbf{X}_0))^{-1} = -\dfrac{1}{25} \begin{bmatrix} 5 & 5 \\ 3 & 8 \end{bmatrix}$; $\mathbf{U}_0 = \mathbf{F}(\mathbf{X}_0) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$;

$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \dfrac{1}{25} \begin{bmatrix} 5 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} u + 5 \\ v - 4 \end{bmatrix}$.

(b) $\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 2xy + y & x^2 + x \\ 2y + y^2 & 2x + 2xy \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}$;

$(\mathbf{F}'(\mathbf{X}_0))^{-1} = \dfrac{1}{6} \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix}$; $\mathbf{U}_0 = \mathbf{F}(\mathbf{X}_0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$;

$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \dfrac{1}{6} \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u - 2 \\ v - 3 \end{bmatrix}$.

(c) $\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 4xy + 3x^2 & 2x^2 & 1 \\ 3x^2 & z & y \\ 1 & 1 & 1 \end{bmatrix}$; $\mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$;

$$(\mathbf{F}'(\mathbf{X}_0))^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \mathbf{U}_0 = \mathbf{F}(\mathbf{X}_0) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix};$$

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u-1 \\ v-1 \\ w-2 \end{bmatrix}.$$

(d) $$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} \cos y \cos z & -x \sin y \cos z & -x \cos y \sin z \\ \sin y \cos z & x \cos y \cos z & -x \sin y \sin z \\ \sin z & 0 & x \cos z \end{bmatrix}; \mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix};$$

$$(\mathbf{F}'(\mathbf{X}_0))^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \mathbf{U}_0 = \mathbf{F}(\mathbf{X}_0) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix};$$

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 1 \\ \pi/2 \\ \pi \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v+1 \\ w \end{bmatrix}.$$

6.3.21. From Exercise 6.2.14**(b)**, $$\mathbf{F}'(r,\theta,\phi) = \begin{bmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\phi & 0 & r\cos\phi \end{bmatrix}$$
and $J\mathbf{F}(r,\theta,\phi) = r^2\cos\phi$. The cofactors of $\mathbf{F}'(r,\theta,\phi)$ are

$$c_{11} = \begin{bmatrix} r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & r\cos\phi \end{bmatrix} = r^2\cos\theta\cos^2\phi;$$

$$c_{12} = -\begin{bmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} = -r\sin\theta;$$

$$c_{13} = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi \\ \sin\phi & 0 \end{bmatrix} = -r\cos\theta\sin\phi\cos\phi;$$

$$c_{21} = -\begin{bmatrix} -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ 0 & r\cos\phi \end{bmatrix} = r^2\sin\theta\cos^2\phi;$$

$$c_{22} = \begin{bmatrix} \cos\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\phi & r\cos\phi \end{bmatrix} = r\cos\theta;$$

$$c_{23} = -\begin{bmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\phi & 0 \end{bmatrix} = -r\sin\theta\sin\phi\cos\phi;$$

$$c_{31} = \begin{bmatrix} -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ r\cos\theta\cos\phi & -r\sin\theta\sin\phi \end{bmatrix} = r^2\cos\phi\sin\phi;$$

$$c_{32} = -\begin{bmatrix} \cos\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\theta\cos\phi & -r\sin\theta\sin\phi \end{bmatrix} = 0;$$

$$c_{33} = \begin{bmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi \end{bmatrix} = r\cos^2\phi;$$

$\mathbf{G}'(x,y,z) = (\mathbf{F}'(r,\theta,\phi))^{-1} =$

$$\frac{1}{r^2\cos\phi}\begin{bmatrix} r^2\cos\theta\cos^2\phi & r^2\sin\theta\cos^2\phi & r^2\cos\phi\sin\phi \\ -r\sin\theta & r\cos\theta & 0 \\ -r\cos\theta\sin\phi\cos\phi & -r\sin\theta\sin\phi\cos\phi & r\cos^2\phi \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta\cos\phi & \sin\theta\cos\phi & \sin\phi \\ -\dfrac{\sin\theta}{r\cos\phi} & \dfrac{\cos\theta}{r\cos\phi} & 0 \\ -\dfrac{1}{r}\cos\theta\sin\phi & -\dfrac{1}{r}\sin\theta\sin\phi & \dfrac{1}{r}\cos\phi \end{bmatrix}.$$

6.3.22. From Exercise 6.2.14**(c)**, $\mathbf{F}'(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $J\mathbf{F}(r,\theta,z) = r$. The cofactors of $\mathbf{F}'(r,\theta,z)$ are

$c_{11} = \begin{bmatrix} r\cos\theta & 0 \\ 0 & 1 \end{bmatrix} = r\cos\theta$; $c_{12} = -\begin{bmatrix} \sin\theta & 0 \\ 0 & 1 \end{bmatrix} = -\sin\theta$;

$c_{13} = \begin{bmatrix} \sin\theta & r\cos\theta \\ 0 & 0 \end{bmatrix} = 0$; $c_{21} = -\begin{bmatrix} -r\sin\theta & 0 \\ 0 & 1 \end{bmatrix} = r\sin\theta$;

$c_{22} = \begin{bmatrix} \cos\theta & 0 \\ 0 & 1 \end{bmatrix} = \cos\theta$; $c_{23} = -\begin{bmatrix} \cos\theta & -r\sin\theta \\ 0 & 0 \end{bmatrix} = 0$;

$c_{31} = \begin{bmatrix} -r\sin\theta & 0 \\ r\cos\theta & 0 \end{bmatrix} = 0$; $c_{32} = -\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} = 0$;

$c_{33} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$;

$$\mathbf{G}'(x,y,z) = (\mathbf{F}'(r,\theta,z))^{-1} = \frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\dfrac{1}{r}\sin\theta & \dfrac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6.3.23. $T = T^0 \cup \partial T$ (since T is closed) and $\mathbf{F}(T) = \mathbf{F}(T)^0 \cup \partial\mathbf{F}(T)$ (since $\mathbf{F}(T)$ is closed, by Exercise 6.2.23). Since $\mathbf{F}$ is regular on T^0, $\mathbf{F}(T^0)$ is open (Theorem 6.3.3; hence, $\mathbf{F}(T^0) \subset \mathbf{F}(T)^0$. Therefore, $T^0 \subset \mathbf{F}^{-1}\left(\mathbf{F}(T)^0\right)$. Since $\mathbf{F}^{-1}$ is one-to-one, this means that $\mathbf{F}^{-1}(\partial\mathbf{F}(T)) \cap T^0 = \emptyset$; hence, $\mathbf{F}^{-1}(\partial\mathbf{F}(T)) \subset \partial T$, so (A) $\partial\mathbf{F}(T) \subset \mathbf{F}(\partial T)$. Since $\mathbf{F}^{-1}$ is also regular on $\mathbf{F}(S)$, and $\mathbf{F}(T)$ is a compact subset of $\mathbf{F}(S)$, we can write (A) with $\mathbf{F}$ and T replaced by $\mathbf{F}^{-1}$ and $\mathbf{F}(T)$. This yields $\partial\left(\mathbf{F}^{-1}(\mathbf{F}(T))\right) \subset \mathbf{F}^{-1}(\partial\mathbf{F}(T))$, so $\partial T \subset \mathbf{F}^{-1}(\partial\mathbf{F}(T))$ and $\mathbf{F}(\partial T) \subset \partial\mathbf{F}(T)$. This and (A) imply that $\mathbf{F}(\partial T) = \partial\mathbf{F}(T)$.

6.4 THE IMPLICIT FUNCTION THEOREM

6.4.1. **(a)**

$$\begin{bmatrix} u \\ v \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} -3 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(b) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = -\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= -\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(c) $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -y + \sin x \\ -x + \sin y \end{bmatrix}$; $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -y + \sin x \\ -x + \sin y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -y + \sin x \\ -x + \sin y \end{bmatrix}.$

(d) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} u + x \\ v + y \\ w + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; since $\begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 13$, $u = -x$, $v = -y$, $z = -w$.

6.4.2. Since N_1 is a neighborhood of $(\mathbf{X}_0, \mathbf{U}_0)$, there is a $\delta > 0$ such that $(\mathbf{X}, \mathbf{U}) \in N_1$ if $|\mathbf{X}-\mathbf{X}_0|^2+|\mathbf{U}-\mathbf{U}_0|^2 < \delta^2$. Therefore, $(\mathbf{X}, \mathbf{U}_0) \in N_1$ if $\mathbf{X} \in N = \left\{\mathbf{X} \in \mathbb{R}^n \,\middle|\, |\mathbf{X} - \mathbf{X}_0| < \delta\right\}$.

6.4.3. Let $\mathbf{X} = (x_{10}, x_{20}, \dots, x_{n0})$, $\mathbf{U} = (u_{10}, u_{20}, \dots, u_{m0})$, and $\mathbf{F} = (f_1, f_2, \dots, f_m)$, where $f_i(\mathbf{X}, \mathbf{U}) = \left(\sum_{j=1}^{n} a_{ij}(x_j - x_{j0})\right)^r - (u_i - u_{i0})^s$, $1 \le i \le m$, where r and s are positive integers and not all $a_{ij} = 0$. Since

$$\mathbf{F}_{\mathrm{U}}(\mathbf{X}, \mathbf{U}) = s \begin{bmatrix} (u_1 - u_{10})^{s-1} & 0 & \cdots & 0 \\ 0 & (u_2 - u_{20})^{s-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (u_m - u_{m0}^{s-2}) \end{bmatrix},$$

$\mathbf{F}_{\mathrm{U}}(\mathbf{X}, \mathbf{U}_0) = \mathbf{0}$ for arbitrary $\mathbf{X}$ if $s > 1$.

(a) If $r = s = 3$, then $\mathbf{F}(\mathbf{X}, \mathbf{U}) = \mathbf{0}$ if and only if $\mathbf{U} = \mathbf{U}_0 + \mathbf{A}(\mathbf{X} - \mathbf{X}_0)$, with $\mathbf{A} = [a_{ij}]$. To see this, note that $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) = 0$ (with α and β real) if and only if $\alpha = \beta$.

(b) If $r = 1$ and $s = 3$, then $\mathbf{F}(\mathbf{X}, \mathbf{U}) = \mathbf{0}$ if and only if

$$u_i = u_{i0} + \left(\sum_{j=1}^{n} a_{ij}(x_j - x_{j0}) \right)^{1/3};$$

$\mathbf{U}$ is a continuous for all $\mathbf{X}$, and differentiable except at $\mathbf{X}_0$.

(c) $r = s = 2$.

6.4.4. $f(x, y, u) = x^2yu + 2xy^2u^3 - 3x^3y^3u^5$; $f_x(x, y, u) = 2xyu + 2y^2u^3 - 9x^2y^3u^5$; $f_x(1, 1, 1) = -5$; $f_y(x, y, u) = x^2u + 4xyu^3 - 9x^3y^2u^5$; $f_y(1, 1, 1) = -4$; $f_u(x, y, u) = x^2y + 6xy^2u^2 - 15x^3y^3u^4$; $f_u(1, 1, 1) = -8$;
$u_x(1, 1) = -\dfrac{f_x(1, 1, 1)}{f_u(1, 1, 1)} = -\dfrac{5}{8}$; $u_y(1, 1) = -\dfrac{f_y(1, 1, 1)}{f_u(1, 1, 1)} = -\dfrac{1}{2}$;

6.4.5. $f(x, y, z, u) = x^2y^5z^2u^5 + 2xy^2u^3 - 3x^3z^2u$;
$f_x(x, y, z, u) = 2xy^5z^2u^5 + 2y^2u^3 - 9x^2z^2u$; $f_x(1, 1, 1, 1) = -5$;
$f_y(x, y, z, u) = 5x^2y^4z^2u^5 + 4xyu^3$; $f_y(1, 1, 1, 1) = 9$;
$f_z(x, y, z, u) = 2x^2y^5zu^5 - 6x^3zu$; $f_z(1, 1, 1, 1) = -4$;
$f_u(x, y, z, u) = 5x^2y^5z^2u^4 + 6xy^2u^2 - 3x^3z^2$; $f_u(1, 1, 1, 1) = 8$;
$u_x(1, 1, 1) = -\dfrac{f_x 1, 1, 1, 1)}{f_u(1, 1, 1, 1)} = \dfrac{5}{8}$; $u_y(1, 1, 1) = -\dfrac{f_y(1, 1, 1, 1)}{f_u(1, 1, 1, 1)} = -\dfrac{9}{8}$;
$u_z(1, 1, 1) = -\dfrac{f_z(1, 1, 1, 1)}{f_u(1, 1, 1, 1)} = \dfrac{1}{2}$.

6.4.6. **(a)** $f(x, y, u) = 2x^2 + y^2 + ue^u$;
$f(1, 2, u(1, 2)) = 2 + 4 + u(1, 2)e^{u(1,2)} = 6$; $u(1, 2) = 0$;
$f_x(x, y, u) = 4x$; $f_x(1, 2, 0) = 4$; $f_y(x, y, u) = 2y$; $f_y(1, 2, 0) = 4$;
$f_u(x, y, u) = e^u(u + 1)$; $f_u(1, 2, 0) = 1$;
$u_x(1, 2) = -\dfrac{f_x(1, 2, 0)}{f_u(1, 2, 0)} = -4$; $u_y(1, 2) = -\dfrac{f_y(1, 2, 0)}{f_u(1, 2, 0)} = -4$.

(b) $f(x, y, u) = u(x + 1) + x(y + 2) + y(u - 2)$;
$f(-1, -2, u(-1, -2)) = -2(u(-1, -2) - 2) = 0$; $u(-1, -2) = 2$;
$f_x(x, y, u) = u + (y + 2)$; $f_x(-1, -2, 2) = 2$;
$f_y(x, y, u) = x + (u - 2)y$; $f_y(-1, -2, 2) = -1$;
$f_u(x, y, u) = (x + 1) + y$; $f_u(-1, -2, 2) = -2$;
$u_x(-1, -2) = -\dfrac{f_x(-1, -2, 2)}{f_u(-1, -2, 2)} = 1$; $u_y(-1, -2) = -\dfrac{f_y(-1, -2, 2)}{f_u(-1, -2, 2)} = -\dfrac{1}{2}$.

(c) $f(x, y, u) = 1 - e^u \sin(x + y)$;

$f(\pi/4, \pi/4, u(\pi/4, \pi/4)) = 1 - e^{u(\pi/4,\pi/4)} = 0$; $u(\pi/4, \pi/4) = 0$;
$f_x(x, y, u) = -e^u \cos(x + y)$; $f_x(\pi/4, \pi/4, 0) = 0$;
$f_y(x, y, u) = -e^u \cos(x + y)$; $f_y(\pi/4, \pi/4, 0) = 0$;
$f_u(x, y, u) = -e^u \sin(x + y)$; $f_u(\pi/4, \pi/4, 0) = 1$;
$u_x(\pi/4, \pi/4) = -\dfrac{f_x(\pi/4, \pi/4, 0)}{f_u(\pi/4, \pi/4, 0)} = 0$; $u_y(\pi/4, \pi/4) = -\dfrac{f_y(\pi/4, \pi/4, 0)}{f_u(\pi/4, \pi/4, 0)} = 0$.
(d) $f(x, y, u) = x \log u + y \log x + u \log y$;
$f(1, 1, u(1, 1)) = \log u(1, 1) = 0$; $u(1, 1) = 1$;
$f_x(x, y, u) = \log u + \dfrac{y}{x}$; $f_x(1, 1, 1) = 1$; $f_y(x, y, u) = \log x + \dfrac{u}{y}$; $f_y(1, 1, 1) = 1$;
$f_u(x, y, u) = \dfrac{x}{u} + \log y$; $f_u(1, 1, 1) = 1$;
$u_x(1, 1) = -\dfrac{f_x(1, 1, 1)}{f_u(1, 1, 1)} = -1$; $u_y(1, 1) = -\dfrac{f_y(1, 1, 1)}{f_u(1, 1, 1)} = -1$.
6.4.7. **(a)** $f(x, y, u) = 2x^2y^4 - 3uxy^3 + u^2x^4y^3$;
$f(1, 1, u(1, 1)) = 2 - 3u(1, 1) + u^2(1, 1) = (u(1, 1) - 1)(u(1, 1) - 2) = 0$;
$u_1(1, 1) = 1, u_2(1, 1) = 2$.
$f_x(x, y, u) = 4xy^4 - 3uy^3 + 4u^2x^3y^3$;
$f_x(1, 1, u_1(1, 1)) = f_x(1, 1, 1) = 5$; $f_x(1, 1, u_2(1, 1)) = f_x(1, 1, 2) = 14$;
$f_y(x, y, u) = 8x^2y^3 - 9uxy^2 + 3u^2x^4y^2$;
$f_y(1, 1, u_1(1, 1)) = f_y(1, 1, 1) = 2$; $f_y(1, 1, u_2(1, 1)) = f_y(1, 1, 2) = 2$;
$f_u(x, y, u) = -3xy^3 + 2ux^4y^3$;
$f_u(1, 1, u_1(1, 1)) = f_u(1, 1, 1) = -1$; $f_u(1, 1, u_2(1, 1)) = f_u(1, 1, 2) = 1$;
$\dfrac{\partial u_1(1, 1)}{\partial x} = -\dfrac{f_x(1, 1, 1)}{f_u(1, 1, 1)} = 5$; $\dfrac{\partial u_2(1, 1)}{\partial x} = -\dfrac{f_x(1, 1, 2)}{f_u(1, 1, 2)} = -14$;
$\dfrac{\partial u_1(1, 1)}{\partial y} = -\dfrac{f_y(1, 1, 1)}{f_u(1, 1, 1)} = 2$; $\dfrac{\partial u_2(1, 1)}{\partial y} = -\dfrac{f_y(1, 1, 2)}{f_u(1, 1, 2)} = -2$.
(b) $f(x, y, u) = \cos u \cos x + \sin u \sin y = 0$;
$f(0, \pi, u(0, \pi)) = \cos u(0, \pi) = 0$; $u_k(0, \pi) = \dfrac{(2k + 1)\pi}{2}$;
$f_x(x, y, u) = -\cos u \sin x$; $f_x(0, \pi, u_k(0, \pi)) = f_x(0, \pi, (2k + 1)\pi/2) = 0$;
$f_y(x, y, u) = \sin u \cos y$; $f_y(0, \pi, u_k(0, \pi)) = f_y(0, \pi, (2k + 1)\pi/2) = (-1)^k$;
$f_u(x, y, u) = -\sin u \cos x + \cos u \sin y = 0$;
$f_u(0, \pi, u_k(0, \pi)) = f_u(0, \pi, (2k + 1)\pi/2) = (-1)^k$;
$\dfrac{\partial u_k(0, \pi)}{\partial x} = -\dfrac{f_x(0, \pi, u_k(0, \pi))}{f_u(0, \pi, u_k(0, \pi))} = 0$;
$\dfrac{\partial u_k(0, \pi)}{\partial y} = -\dfrac{f_y(0, \pi, u_k(0, \pi))}{f_u(0, \pi, u_k(0, \pi))} = -1$.

6.4.8. Let $\mathbf{X}=(x,y,z)$, $\mathbf{X_0}=(1,\frac{1}{2},-1)$, $\mathbf{U}=(u,v)$ and $\mathbf{U_0}=(-2,1)$. Then

$$\mathbf{F}(\mathbf{X},\mathbf{U})=\begin{bmatrix} x^2+4y^2+z^2-2u^2+v^2+4 \\ (x+z)^2+u-v+3 \end{bmatrix}$$

and $\mathbf{F}(\mathbf{X_0},\mathbf{U_0})=\mathbf{0}$.

$\mathbf{F_X}(\mathbf{X},\mathbf{U})=\begin{bmatrix} 2x & 8y & 2z \\ x+z & 0 & x+z \end{bmatrix}$; $\mathbf{F_X}(\mathbf{X_0},\mathbf{U_0})=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{F_U}(\mathbf{X},\mathbf{U})=\begin{bmatrix} -4u & 2v \\ 1 & -1 \end{bmatrix}$; $\mathbf{F_U}(\mathbf{X_0},\mathbf{U_0})=\begin{bmatrix} 8 & 2 \\ 1 & -1 \end{bmatrix}$; $(\mathbf{F_U}(\mathbf{X_0},\mathbf{U_0}))^{-1}=\frac{1}{10}\begin{bmatrix} 1 & 2 \\ 1 & -8 \end{bmatrix}$;

$\mathbf{U}'(\mathbf{X_0})=-(\mathbf{F_U}(\mathbf{X_0},\mathbf{U_0}))^{-1}\mathbf{F_X}(\mathbf{X_0},\mathbf{U_0})=-\frac{1}{10}\begin{bmatrix} 1 & 2 \\ 1 & -8 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}=\frac{1}{5}\begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \end{bmatrix}.$

6.4.9.

$$\begin{aligned} (1+4u)u'+(2v+2)v'+2x-1&=0 \\ (xv+e^u\sin(v+x))u'+(xu+e^u\cos(v+x))v'+uv+e^u\cos(v+x)&=0; \end{aligned}$$

$u'(0)+2v'(0)=1$; $v'(0)=-1$; $u'(0)=3$.

6.4.10. Let $\mathbf{X}=(x,y)$, $\mathbf{X_0}=(1,-1)$, $\mathbf{U}=(u,v,w)$ and $\mathbf{U_0}=(1,2,0)$. Then

$$\mathbf{F}(\mathbf{X},\mathbf{U})=\begin{bmatrix} x^2y+xy^2+u^2-(v+w)^2+3 \\ e^{x+y}-u-v-w+2 \\ (x+y)^2+u+v+w^2-3 \end{bmatrix}$$

and $\mathbf{F}(\mathbf{X_0},\mathbf{U_0})=\mathbf{0}$.

$\mathbf{F_X}(\mathbf{X},\mathbf{U})=\begin{bmatrix} 2xy+y^2 & x^2+2xy \\ e^{x+y} & e^{x+y} \\ 2(x+y) & 2(x+y) \end{bmatrix}$; $\mathbf{F_X}(\mathbf{X_0},\mathbf{U_0})=\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$;

$\mathbf{F_U}(\mathbf{X},\mathbf{U})=\begin{bmatrix} 2u & -2(v+w) & -2(v+w) \\ -1 & -1 & -1 \\ 1 & 1 & 2w \end{bmatrix}$; $\mathbf{F_U}(\mathbf{X_0},\mathbf{U_0})=\begin{bmatrix} 2 & -4 & -4 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$;

$(\mathbf{F_U}(\mathbf{X_0},\mathbf{U_0}))^{-1}=\frac{1}{6}\begin{bmatrix} 1 & -4 & 0 \\ -1 & 4 & 6 \\ 0 & -6 & -6 \end{bmatrix}$;

$$\begin{aligned} \mathbf{U}'(\mathbf{X_0})&=-(\mathbf{F_U}(\mathbf{X_0},\mathbf{U_0}))^{-1}\mathbf{F_X}(\mathbf{X_0},\mathbf{U_0}) \\ &=-\frac{1}{6}\begin{bmatrix} 1 & -4 & 0 \\ -1 & 4 & 6 \\ 0 & -6 & -6 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 5 & 5 \\ -5 & -5 \\ 6 & 6 \end{bmatrix}. \end{aligned}$$

6.4.11. Let $\mathbf{X}=(x,y)$, $\mathbf{X_0}=(1,1)$, and $\mathbf{U}=(u,v)$. Then

$$\mathbf{F}(\mathbf{X},\mathbf{U})=\begin{bmatrix} xyu-4yu+9xv \\ 2xy-3y^2+v^2 \end{bmatrix}$$

and $\mathbf{F}(\mathbf{X}_0, \mathbf{U}_0) = \mathbf{0}$ if $u_0 = 3v_0$ and $v_0^2 = 1$, which is equivalent to $(u_0, v_0) = \pm(3, 1)$. Let $\mathbf{U}_1(\mathbf{X}_0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{U}_2(\mathbf{X}_0) = -\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$\mathbf{F}_\mathbf{X}(\mathbf{X}, \mathbf{U}) = \begin{bmatrix} yu + 9v & (x-4)u \\ 2y & 2x - 6y \end{bmatrix}$; $\mathbf{F}_\mathbf{X}(\mathbf{X}_0, \mathbf{U}_1(\mathbf{X}_0)) = \begin{bmatrix} 12 & -9 \\ 2 & -4 \end{bmatrix}$;

$\mathbf{F}_\mathbf{X}(\mathbf{X}_0, \mathbf{U}_2(\mathbf{X}_0)) = \begin{bmatrix} -12 & 9 \\ 2 & -4 \end{bmatrix}$; $\mathbf{F}_\mathbf{U}(\mathbf{X}, \mathbf{U}) = \begin{bmatrix} (x-4)y & 9x \\ 0 & 2v \end{bmatrix}$;

$\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_1(\mathbf{X}_0)) = \begin{bmatrix} -3 & 9 \\ 0 & 2 \end{bmatrix}$; $\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_2(\mathbf{X}_0)) = \begin{bmatrix} -3 & 9 \\ 0 & -2 \end{bmatrix}$;

$(\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_1(\mathbf{X}_0)))^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 9 \\ 0 & 3 \end{bmatrix}$; $(\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_2(\mathbf{X}_0)))^{-1} = -\frac{1}{6}\begin{bmatrix} 2 & 9 \\ 0 & 3 \end{bmatrix}$;

$$\begin{aligned} \mathbf{U}_1'(\mathbf{X}_0) &= -(\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_1(\mathbf{X}_0)))^{-1}\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_1(\mathbf{X}_0)) \\ &= -\frac{1}{6}\begin{bmatrix} -2 & 9 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 12 & -9 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}; \end{aligned}$$

$$\begin{aligned} \mathbf{U}_2'(\mathbf{X}_0) &= -(\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_2(\mathbf{X}_0)))^{-1}\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_0) \\ &= \frac{1}{6}\begin{bmatrix} 2 & 9 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -12 & 9 \\ 2 & -4 \end{bmatrix} = -\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}. \end{aligned}$$

6.4.12. Differentiate the three equations with respect to x, using the chain rule:

$$\begin{aligned} e^x \cos y - (e^x \sin u)u_x + (e^v \cos w)v_x - (e^v \sin w)w_x + 1 &= 0 \\ e^x \sin y + (e^z \cos u)u_x + (e^v \cos w)v_x - (e^v \sin w)w_x &= 0 \\ e^x \tan y + (e^z \sec^2 u)u_x + (e^v \tan w)v_x + (e^v \sec^2 w)w_x &= 0. \end{aligned}$$

Setting $(x, y, z) = (0, 0, 0)$ yields $v_x(0, 0, 0) = -2$, $u_x(0, 0, 0) + v_x(0, 0, 0) = 0$, $u_x(0, 0, 0) + w_x(0, 0, 0) = 0$; therefore, $u_x(0, 0, 0) = 2$ and $v_x(0, 0, 0) = w_x(0, 0, 0) = -2$.

6.4.14. If ξ_1 and ξ_2 are any two of the five variables and η_1, η_2, and η_3 are the rest, the system can be written in the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This can be solved for $\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ if and only if $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$. We list the possibilities:

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} z \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; since $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$, this system does not determine (x, y) as a function of (z, u, v).

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}+\begin{bmatrix} 2 & 1 & 6 \\ 4 & 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} x \\ z \end{bmatrix}=-\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 & 6 \\ 4 & 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}=\frac{1}{5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 6 \\ 4 & 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}$$

$$=-\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y \\ u \\ v \end{bmatrix}; \quad x=-2y-u, \quad z=-2v;$$

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}+\begin{bmatrix} 2 & 3 & 6 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \\ v \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$; since $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}=0$, this system does not determine (x,u) as a function of (y,z,v).

$$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}+\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \\ u \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} x \\ v \end{bmatrix}=-\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \\ u \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 2 & -6 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \\ u \end{bmatrix}$$

$$=-\frac{1}{2}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \\ u \end{bmatrix}; \quad x=-2y-u, \quad v=-\frac{z}{2}.$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}+\begin{bmatrix} 1 & 1 & 6 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ u \\ v \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} y \\ z \end{bmatrix}=-\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 6 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ u \\ v \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 6 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ u \\ v \end{bmatrix}$$

$$=-\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ u \\ v \end{bmatrix}; \quad y=-\frac{1}{2}(x+u), \quad z=-2v.$$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix}+\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ z \\ v \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$; since $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}=0$, the system does not determine (y,u) as a function of (x,z,v).

$$\begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix}+\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ z \\ u \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} y \\ v \end{bmatrix} = -\begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 2 & -6 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ u \end{bmatrix}; \quad y = -\frac{1}{2}(x+u), \quad v = -\frac{z}{2}.$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} + \begin{bmatrix} 1 & 2 & 6 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} z \\ u \end{bmatrix} = -\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 6 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \end{bmatrix} =$$

$$-\begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \end{bmatrix}; z = -2v, u = -x - 2y.$$

$\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; since $\begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} = 0$, this system does not determine (z, v) as a function of (x, y, u).

$$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = -\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & -6 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad u = -x - 2y, \quad v = -\frac{z}{2}.$$

6.4.15. Let

$$f(x, y, z, u, v) = x^2 + 4y^2 + z^2 - 2u^2 + v^2 + 4$$
$$g(x, y, z, u, v) = (x + z)^2 + u - v + 3$$

and $\mathbf{P}_0 = (1, \frac{1}{2}, -1, -2, 1)$.

$$\frac{\partial(f, g)}{\partial(y, v)} = \begin{vmatrix} 8y & 2v \\ 0 & -1 \end{vmatrix}; \left.\frac{\partial(f, g)}{\partial(y, v)}\right|_{\mathbf{P}_0} = \begin{vmatrix} 4 & 2 \\ 0 & -1 \end{vmatrix} = -4;$$

$$\frac{\partial(f, g)}{\partial(x, v)} = \begin{vmatrix} 2x & 2v \\ 2(x+z) & -1 \end{vmatrix}; \left.\frac{\partial(f, g)}{\partial(x, v)}\right|_{\mathbf{P}_0} = \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} = -2;$$

$$y_x(1,-1,-2) = -\frac{\left.\dfrac{\partial(f,g)}{\partial(x,v)}\right|_{\mathbf{P}_0}}{\left.\dfrac{\partial(f,g)}{\partial(y,v)}\right|_{\mathbf{P}_0}} = -\frac{-2}{-4} = -\frac{1}{2};$$

$$\frac{\partial(f,g)}{\partial(y,u)} = \begin{vmatrix} 8y & -4u \\ 0 & 1 \end{vmatrix}; \left.\frac{\partial(f,g)}{\partial(y,u)}\right|_{\mathbf{P}_0} = \begin{vmatrix} 4 & 8 \\ 0 & 1 \end{vmatrix} = 4;$$

$$v_u(1,-1,-2) = -\frac{\left.\dfrac{\partial(f,g)}{\partial(y,u)}\right|_{\mathbf{P}_0}}{\left.\dfrac{\partial(f,g)}{\partial(y,v)}\right|_{\mathbf{P}_0}} = -\frac{4}{-4} = 1.$$

6.4.16. Apply Theorem 6.4.1 with $\mathbf{X} = (w, y)$, $\mathbf{X_0} = (0, -1)$, $\mathbf{U} = (u, v, x)$ and $\mathbf{U_0} = (1, 2, 1)$. Then

$$\mathbf{F}(\mathbf{X}, \mathbf{U}) = \begin{bmatrix} x^2y + xy^2 + u^2 - (v+w)^2 + 3 \\ e^{x+y} - u - v - w + 2 \\ (x+y)^2 + u + v + w^2 - 3 \end{bmatrix}$$

and $\mathbf{F}(\mathbf{X_0}, \mathbf{U_0}) = \mathbf{0}$.

$$\mathbf{F_X}(\mathbf{X}, \mathbf{U}) = \begin{bmatrix} -2(v+w) & x^2 + 2xy \\ -1 & e^{x+y} \\ 2w & 2(x+y) \end{bmatrix}; \mathbf{F_X}(\mathbf{X_0}, \mathbf{U_0}) = \begin{bmatrix} -4 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix};$$

$$\mathbf{F_U}(\mathbf{X}, \mathbf{U}) = \begin{bmatrix} 2u & -2(v+w) & 2xy + y^2 \\ -1 & -1 & e^{x+y} \\ 1 & 1 & 2(x+y) \end{bmatrix}; \quad \mathbf{F_U}(\mathbf{X_0}, \mathbf{U_0}) = \begin{bmatrix} 2 & -4 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix};$$

$$(\mathbf{F_U}(\mathbf{X_0}, \mathbf{U_0}))^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 5 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix};$$

$$\mathbf{U}'(\mathbf{X_0}) = -(\mathbf{F_U}(\mathbf{X_0}, \mathbf{U_0}))^{-1}\mathbf{F_X}(\mathbf{X_0}, \mathbf{U_0}) =$$

$$-\frac{1}{6}\begin{bmatrix} 1 & 1 & 5 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -5 & 0 \\ 5 & 0 \\ -6 & 6 \end{bmatrix};$$

$u_w(0, -1) = \frac{5}{6}$, $u_y(0, -1) = 0$, $v_w(0, -1) = -\frac{5}{6}$, $v_y(0, -1) = 0$, $x_w(0, -1) = 1$, $x_y(0, -1) = -1$.

6.4.17. See the solution of Exercise 6.3.11.

6.4.18. Let $\mathbf{X} = (x, y)$, $\mathbf{X_0} = (1, 1)$, $\mathbf{U} = (u, v)$ and $\mathbf{U_0} = (0, 1)$. Then

$$\mathbf{F}(\mathbf{X}, \mathbf{U}) = \begin{bmatrix} x^2 + y^2 + u^2 + v^2 - 3 \\ x \;\; + y \;\; + u \;\; + v \;\; - 3, \end{bmatrix}$$

and $\mathbf{F}(\mathbf{X}_0, \mathbf{U}_0) = \mathbf{0}$.

$\mathbf{F}_{\mathrm{X}}(\mathbf{X},\mathbf{U}) = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix}$; $\mathbf{F}_{\mathrm{U}}(\mathbf{X},\mathbf{U}) = \begin{bmatrix} 2u & 2v \\ 1 & 1 \end{bmatrix}$;

$(\mathbf{F}_{\mathrm{U}}(\mathbf{X},\mathbf{U}))^{-1} = \dfrac{1}{2(u-v)}\begin{bmatrix} 1 & -2v \\ -1 & 2u \end{bmatrix}$;

$\mathbf{U}'(\mathbf{X}) = -(\mathbf{F}_{\mathrm{U}}(\mathbf{X},\mathbf{U}))^{-1}\mathbf{F}_{\mathrm{X}}(\mathbf{X},\mathbf{U}) = \dfrac{1}{u-v}\begin{bmatrix} v-x & v-y \\ x-u & y-u \end{bmatrix}$;

$u_x = \dfrac{v-x}{u-v}$; $u_x(1,1) = 0$; $u_y = \dfrac{v-y}{u-v}$; $u_y(1,1) = 0$;

$v_x = \dfrac{x-u}{u-v}$; $v_x(1,1) = -1$; $v_y = \dfrac{y-u}{u-v}$; $v_y(1,1) = -1$;

$u_{xx} = \dfrac{(u-v)(v_x-1)-(v-x)(u_x-v_x)}{(u-v)^2}$; $u_{xx}(1,1) = 2$;

$u_{xy} = \dfrac{(u-v)v_y-(v-x)(u_y-v_y)}{(u-v)^2}$; $u_{xy}(1,1) = 1$;

$u_{yy} = \dfrac{(u-v)(v_y-1)-(v-y)(u_y-v_y)}{(u-v)^2}$; $u_{yy}(1,1) = 2$;

$v_{xx} = \dfrac{(u-v)(1-u_x)-(x-u)(u_x-v_x)}{(u-v)^2}$; $v_{xx}(1,1) = -2$;

$v_{xy} = \dfrac{-(u-v)u_y-(x-u)(u_y-v_y)}{(u-v)^2}$; $v_{xy}(1,1) = -1$;

$v_{yy} = \dfrac{(u-v)(1-u_y)-(y-u)(u_y-v_y)}{(u-v)^2}$; $v_{yy}(1,1) = -2$;

6.4.19. Let $\mathbf{X} = (x,y)$, $\mathbf{X}_0 = (1,-1)$, $\mathbf{U} = (u,v)$ and $\mathbf{U}_0 = (-1,1)$. Then

$$\mathbf{F}(\mathbf{X},\mathbf{U}) = \begin{bmatrix} u^2-v^2-x+y+2 \\ 2uv-x-y+2 \end{bmatrix}$$

and $\mathbf{F}(\mathbf{X}_0, \mathbf{U}_0) = \mathbf{0}$.

$\mathbf{F}_{\mathrm{X}}(\mathbf{X},\mathbf{U}) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$; $\mathbf{F}_{\mathrm{U}}(\mathbf{X},\mathbf{U}) = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$;

$(\mathbf{F}_{\mathrm{U}}(\mathbf{X},\mathbf{U}))^{-1} = \dfrac{1}{2(u^2+v^2)}\begin{bmatrix} u & v \\ -v & u \end{bmatrix}$;

$\mathbf{U}'(\mathbf{X}) = -(\mathbf{F}_{\mathrm{U}}(\mathbf{X},\mathbf{U}))^{-1}\mathbf{F}_{\mathrm{X}}(\mathbf{X},\mathbf{U}) = \dfrac{1}{2(u^2+v^2)}\begin{bmatrix} u+v & v-u \\ u-v & u+v \end{bmatrix}$;

$u_x = \dfrac{u+v}{2(u^2+v^2)}$; $u_x(1,-1) = 0$; $u_y = \dfrac{v-u}{2(u^2+v^2)}$; $u_y(1,-1) = \dfrac{1}{2}$;

$v_x = \dfrac{u-v}{2(u^2+v^2)}$; $v_x(1,-1) = -\dfrac{1}{2}$; $v_y = \dfrac{u+v}{2(u^2+v^2)}$; $v_y(1,-1) = 0$;

$u_{xx} = \dfrac{u_x+v_x}{2(u^2+v^2)} - \dfrac{(u+v)(uu_x+vv_x)}{(u^2+v^2)^2}$; $u_{xx}(1,-1) = -\dfrac{1}{8}$;

$$u_{xy} = \frac{u_y + v_y}{2(u^2 + v^2)} - \frac{(u + v)(uu_y + vv_y)}{(u^2 + v^2)^2}; \; u_{xy}(1, -1) = \frac{1}{8};$$
$$u_{yy} = \frac{v_y - u_y}{2(u^2 + v^2)} - \frac{(v - u)(uu_y + vv_y)}{(u^2 + v^2)^2}; \; u_{yy}(1, -1) = \frac{1}{8};$$
$$v_{xx} = \frac{u_x - v_x}{2(u^2 + v^2)} - \frac{(u - v)(uu_x + vv_x)}{(u^2 + v^2)^2}; \; v_{xx}(1, -1) = -\frac{1}{8};$$
$$v_{xy} = \frac{u_x + v_x}{2(u^2 + v^2)} - \frac{(u + v)(uu_x + vv_x)}{(u^2 + v^2)^2}; \; v_{xy}(1, -1) = -\frac{1}{8};$$
$$v_{yy} = \frac{u_y + v_y}{2(u^2 + v^2)} - \frac{(u + v)(uu_y + vv_y)}{(u^2 + v^2)^2}; \; v_{yy}(1, -1) = \frac{1}{8}.$$

6.4.20. Differentiating the identity

$$\phi(f_1(\mathbf{X}), f_2(\mathbf{X}), \dots, f_n(\mathbf{X})) = 0, \quad \mathbf{X} \in S,$$

with respect to $x_1, x_2, \dots, x_n$ yields

$$\sum_{j=1}^{n} \frac{\partial \phi}{\partial u_j}(\mathbf{F}(\mathbf{X})) \frac{\partial f_j}{\partial x_i}(\mathbf{X}) = 0, \quad 1 \le i \le n, \; \mathbf{X} \in S. \tag{A}$$

Since

$$\sum_{j=1}^{n} \phi_{u_j}^2(\mathbf{F}(\mathbf{X}) > 0, \quad \mathbf{X} \in S,$$

(A) implies that the system

$$\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}(\mathbf{X}) \xi_j = 0, \quad 1 \le i \le n, \; \mathbf{X} \in S.$$

has the nontrivial solution $(\xi_1, \dots, \xi_n) = (\phi_{u_1}(\mathbf{F}(\mathbf{X})), \dots, \phi_{u_n}(\mathbf{F}(\mathbf{X})))$ for every $\mathbf{X} \in S$. Now Theorem 6.1.15 implies the conclusion.

CHAPTER 7

INTEGRALS OF FUNCTIONS OF SEVERAL VARIABLES

7.1 DEFINITION AND EXISTENCE OF THE MULTIPLE INTEGRAL

7.1.1. Suppose, for example, that $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ with $a_1 = b_1$. Then every partition $\mathbf{P}$ of R is of the form $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$, where $v(R_j) = 0$, $1 \le j \le k$. Therefore, every Riemann sum of f over R equals zero, so $\int_R f(\mathbf{X})\,d\mathbf{X} = 0$.

7.1.2. **(a)** Let P_1 and P_2 be partitions of $[0, 2]$ and $[1, 3]$; thus,

$$P_1 : 0 = x_0 < x_1 < \cdots < x_r = 2 \quad \text{and} \quad P_2 : 1 = y_0 < y_1 < \cdots < y_s = 3.$$

A typical Riemann sum of f over $\mathbf{P} = P_1 \times P_2$ is given by

$$\sigma = \sum_{i=1}^{r} \sum_{j=1}^{s} (3\xi_{ij} + 2\eta_{ij})(x_i - x_{i-1})(y_j - y_{j-1}) \tag{A}$$

where

$$x_{i-1} \le \xi_{ij} \le x_i \quad \text{and} \quad y_{j-1} \le \eta_{ij} \le y_j. \tag{B}$$

The midpoints of $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ are

$$\overline{x}_i = \frac{x_i + x_{i-1}}{2} \quad \text{and} \quad \overline{y}_j = \frac{y_j + y_{j-1}}{2}, \tag{C}$$

and (B) implies that

$$|\xi_{ij} - \overline{x}_i| \le \frac{x_i - x_{i-1}}{2} \le \frac{\|P_1\|}{2} \le \frac{\|\mathbf{P}\|}{2} \tag{D}$$

and

$$|\eta_{ij} - \overline{y}_j| \le \frac{y_j - y_{j-1}}{2} \le \frac{\|P_2\|}{2} \le \frac{\|\mathbf{P}\|}{2}. \tag{E}$$

Now we rewrite (A) as

$$\begin{aligned}\sigma = &\sum_{i=1}^{r}\sum_{j=1}^{s}(3\overline{x}_i + 2\overline{y}_j)(x_i - x_{i-1})(y_j - y_{j-1}) \\ &+\sum_{i=1}^{r}\sum_{j=1}^{s}\left[3(\xi_{ij} - \overline{x}_i) + 2(\eta_{ij} - \overline{y}_j)\right](x_i - x_{i-1})(y_j - y_{j-1}).\end{aligned} \tag{F}$$

To find $\int_R f(x,y)\,d(x,y)$ from this, we recall that

$$\sum_{i=1}^{r}(x_i - x_{i-1}) = 2, \quad \sum_{j=1}^{s}(y_j - y_{j-1}) = 2 \tag{G}$$

(Example 3.1.1), and

$$\sum_{i=1}^{r}(x_i^2 - x_{i-1}^2) = 4, \quad \sum_{j=1}^{s}(y_j^2 - y_{j-1}^2) = 8 \tag{H}$$

(Example 3.1.2).

Because of (D), (E), and (G), the absolute value of the second sum in (F) does not exceed

$$\frac{5}{2}\|\mathbf{P}\|\sum_{j=1}^{r}\sum_{j=1}^{s}(x_i - x_{i-1})(y_j - y_{j-1}) = 10\|P\|,$$

so (F) implies that

$$\left|\sigma - \sum_{i=1}^{r}\sum_{j=1}^{s}(3\overline{x}_i + 2\overline{y}_j)(x_i - x_{i-1})(y_j - y_{j-1})\right| \le 10\|\mathbf{P}\|. \tag{I}$$

Now

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j=1}^{s}\overline{x}_i(x_i - x_{i-1})(y_j - y_{j-1}) &= \left[\sum_{i=1}^{r}\overline{x}_i(x_i - x_{i-1})\right]\left[\sum_{j=1}^{s}(y_j - y_{j-1})\right] \\ &= 2\sum_{i=1}^{r}\overline{x}_i(x_i - x_{i-1}) \qquad \text{(from (G))} \\ &= \sum_{i=1}^{r}(x_i^2 - x_{i-1}^2) \qquad \text{(from (C))} \\ &= 4 \qquad \text{(from (H))}.\end{aligned}$$

Similarly, $\sum_{i=1}^{r}\sum_{j=1}^{s}\overline{y}_j(x_i-x_{i-1})(y_j-y_{j-1}) = 8$. Therefore, (I) can be written as $|\sigma-28| \le 10\|\mathbf{P}\|$. Since the right side can be made as small as we wish by choosing $\|\mathbf{P}\|$ sufficiently small, $\int_R (x+y)\,d(x,y) = 28$.

(b) Let P_1 and P_2 be partitions of $[0,1]$; thus,

$$P_1 : 0 = x_0 < x_1 < \cdots < x_r = 1 \quad \text{and} \quad P_2 : 0 = y_0 < y_1 < \cdots < y_s = 1.$$

A typical Riemann sum of f over $\mathbf{P} = P_1 \times P_2$ is given by

$$\sigma = \sum_{i=1}^{r}\sum_{j=1}^{s}\xi_{ij}\eta_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) \tag{A}$$

where

$$x_{i-1} \le \xi_{ij} \le x_i \quad \text{and} \quad y_{j-1} \le \eta_{ij} \le y_j. \tag{B}$$

The midpoints of $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ are

$$\overline{x}_i = \frac{x_i + x_{i-1}}{2} \quad \text{and} \quad \overline{y}_j = \frac{y_j + y_{j-1}}{2}, \tag{C}$$

and (B) implies that

$$|\xi_{ij} - \overline{x}_i| \le \frac{x_i - x_{i-1}}{2} \le \frac{\|P_1\|}{2} \le \frac{\|\mathbf{P}\|}{2} \tag{D}$$

and

$$|\eta_{ij} - \overline{y}_j| \le \frac{y_j - y_{j-1}}{2} \le \frac{\|P_2\|}{2} \le \frac{\|\mathbf{P}\|}{2}. \tag{E}$$

Since

$$\begin{aligned}\xi_{ij}\eta_{ij} &= [\overline{x}_i + (\xi_{ij} - \overline{x}_i)][\overline{y}_j + (\eta_{ij} - \overline{y}_j)]\\ &= \overline{x}_i\overline{y}_j + \overline{x}_i(\eta_{ij} - \overline{y}_j) + \overline{y}_j(\xi_{ij} - \overline{x}_i) + (\xi_{ij} - \overline{x}_i)(\eta_{ij} - \overline{y}_j),\end{aligned}$$

we can rewrite (A) as $\sigma = S_1 + S_2 + S_3 + S_4$ where

$$\begin{aligned}S_1 &= \left(\sum_{i=1}^{r}\overline{x}_i(x_i - x_{i-1})\right)\left(\sum_{j=1}^{s}\overline{y}_j(y_j - y_{j-1})\right)\\ &= \frac{1}{4}\left(\sum_{i=1}^{r}(x_i^2 - x_{i-1}^2)\right)\left(\sum_{j=1}^{s}(y_j^2 - y_{j-1}^2)\right) = \frac{1}{4},\end{aligned} \tag{F}$$

$$S_2 = \sum_{i=1}^{r}\sum_{j=1}^{s}\overline{x}_i(\eta_{ij} - \overline{y}_j)(x_i - x_{i-1})(y_j - y_{j-1}), \tag{G}$$

$$S_3 = \sum_{i=1}^{r}\sum_{j=1}^{s}\overline{y}_j(\xi_{ij} - \overline{x}_i)(x_i - x_{i-1})(y_j - y_{j-1}), \tag{H}$$

and

$$S_4 = \sum_{i=1}^{r}\sum_{j=1}^{s}(\xi_{ij} - \overline{x}_i)(\eta_{ij} - \overline{y}_j)(x_i - x_{i-1})(y_j - y_{j-1}). \tag{I}$$

From (E) and (G),

$$|S_2| \le \frac{\|P\|}{2}\left(\sum_{i=1}^{r}\overline{x}_i(x_i - x_{i-1})\right)\left(\sum_{j=1}^{s}(y_j - y_{j-1})\right) = \frac{\|P\|}{4}\sum_{i=1}^{r}(x_i^2 - x_{i-1}^2) = \frac{\|P\|}{4}.$$

From (D) and (H),

$$|S_3| \le \frac{\|P\|}{2}\left(\sum_{i=1}^{r}(x_i - x_{i-1})\right)\left(\sum_{j=1}^{s}\overline{y}_j(y_j - y_{j-1})\right) = \frac{\|P\|}{4}\sum_{j=1}^{s}(y_j^2 - y_{j-1}^2) = \frac{\|P\|}{4}.$$

From (D), (E), and (I),

$$|S_4| \le \frac{\|P\|^2}{4}\left(\sum_{i=1}^{r}(x_i - x_{i-1})\right)\left(\sum_{j=1}^{s}(y_j - y_{j-1})\right) = \frac{\|P\|^2}{4}.$$

Therefore, $\left|\sigma - \frac{1}{4}\right| \le \frac{\|P\|}{2} + \frac{\|P\|^2}{4}$. Since the right side can be made as small as we wish by choosing $\|\mathbf{P}\|$ sufficiently small, $\int_R xy\, d(x,y) = \frac{1}{4}$.

7.1.3. The given σ is not a typical Riemann sum. The correct form is

$$\sigma = \sum_{i=1}^{r}\sum_{j=1}^{s} f(\xi_{ij})g(\eta_{ij})(x_i - x_{i-1})(y_j - y_{j-1}),$$

where $x_{i-1} \le \xi_{ij} \le x_i$ and $y_{j-1} \le \eta_{ij} \le y_j$.

7.1.4. Let

$$P_1 : a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad P_2 : c = y_0 < y_1 < \cdots < y_s = d$$

be partitions of $[a,b]$ and $[c,d]$, and $\mathbf{P} = P_1 \times P_2$. Then a typical Riemann sum of f over $\mathbf{P}$ is of the form $\sigma = \sum_{i=1}^{r}\sum_{j=1}^{s} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1})$, which can be interpreted as the sum of the volumes of parallelepipeds in $\mathbb{R}^3$ with bases of areas $(x_i - x_{i-1})(y_j - y_{j-1})$ and heights $f(\xi_{ij}, \eta_{ij})$, $1 \le i \le r$, $1 \le j \le s$.

7.1.5. **(a)** If $\mathbf{P} = \{R_1, R_12 \dots, R_k\}$, then $S(\mathbf{P}) = \sum_{j=1}^{k} M_j V(R_j)$, where $M_j = \sup\left\{f(\mathbf{X}) \,\middle|\, x \in R_j\right\}$.

An arbitrary Riemann sum of f over $\mathbf{P}$ is of the form $\sigma = \sum_{j=1}^{k} f(\mathbf{X}_j)V(R_j)$, where $\mathbf{X}_j \in R_j$. Since $f(\mathbf{X}_j) \le M_j, \sigma \le S(\mathbf{P})$.

Now let $\epsilon > 0$ and choose $\overline{\mathbf{X}}_j$ in R_j so that $f(\overline{\mathbf{X}}_j) > M_j - \dfrac{\epsilon}{nV(R_j)}$, $1 \le j \le k$. The Riemann sum produced in this way is $\overline{\sigma} = \sum_{j=1}^{k} f(\overline{\mathbf{X}}_j)V(R_j) > \sum_{j=1}^{k} \left[M_j - \dfrac{\epsilon}{kV(R_j)})\right] V(R_j) = S(\mathbf{P}) - \epsilon$. Now Theorem 1.1.3 implies that $S(\mathbf{P})$ is the supremum of the set of Riemann sums of f over $\mathbf{P}$.

(b) Let $\mathbf{P}$ be as in **(a)**. Then $s(\mathbf{P}) = \sum_{j=1}^{k} m_j V(R_j)$, where $m_j = \inf\left\{f(\mathbf{X}) \,\middle|\, x \in R_j\right\}$. An arbitrary Reimann sum of f over $\mathbf{P}$ is of the form $\sigma = \sum_{j=1}^{k} f(\mathbf{X}_j)V(R_j)$ where $\mathbf{X}_j \in R_j$. Since $f(\mathbf{X}_j) \ge m_j, \sigma \ge s(\mathbf{P})$.

Now let $\epsilon > 0$ and choose $\overline{\mathbf{X}}_j$ in R_j so that $f(\overline{\mathbf{X}}_j) < m_j + \dfrac{\epsilon}{kV(R_j)}$, $1 \le j \le k$. The Riemann sum produced in this way is $\overline{\sigma} = \sum_{j=1}^{k} f(\mathbf{X}_j)V(R_j) > \sum_{j=1}^{k} \left[m_j + \dfrac{\epsilon}{kV(R_j)})\right] V(R_j) = s(\mathbf{P}) + \epsilon$. Now Theorem 1.1.8 implies that $s(\mathbf{P})$ is the infimum of the set of Riemann sums of f over $\mathbf{P}$.

7.1.5. Let $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ be a partition of R. An arbitrary Riemann sum of f over $\mathbf{P}$ is of the form $\sigma = \sum_{j=1}^{n} f(\mathbf{X}_j)V(R_j)$ where $\mathbf{X}_j \in R_j$.

(a) $S(\mathbf{P}) = \sum_{j=1}^{k} M_j V(R_j)$ where $M_j = \sup\left\{f(\mathbf{X}) \,\middle|\, \mathbf{X} \in R_j\right\}$. Since $f(\mathbf{X}_j) \le M_j$, $\sigma \le S(\mathbf{P})$. If $\epsilon > 0$ choose $\overline{\mathbf{X}}_j$ in R_j so that $f(\overline{\mathbf{X}}_j) > M_j - \dfrac{\epsilon}{kV(R_j)}$. The Riemann sum produced in this way is

$$\overline{\sigma} = \sum_{j=1}^{k} f(\overline{\mathbf{X}}_j)V(R_j) > \sum_{j=1}^{k} \left[M_j - \frac{\epsilon}{kV(R_j)}\right] V(R_j) = S(\mathbf{P}) - \epsilon.$$

Now Theorem 1.1.3 implies that $S(\mathbf{P})$ is the supremum of the set of Riemann sums of f over $\mathbf{P}$.

(b) $s(\mathbf{P}) = \displaystyle\sum_{j=1}^{k} m_j V(R_j)$ where $m_j = \inf\left\{f(\mathbf{X}) \,\middle|\, \mathbf{X} \in R_j\right\}$. Since $f(\mathbf{X}_j) \ge m_j$, $\sigma \ge s(\mathbf{P})$. If $\epsilon > 0$ choose $\overline{\mathbf{X}}_j$ in R_j so that $f(\overline{\mathbf{X}}_j) < m_j + \dfrac{\epsilon}{kV(R_j)}$. The Riemann sum produced in this way is

$$\overline{\sigma} = \sum_{j=1}^{k} f(\overline{\mathbf{X}}_j)V(R_j) < \sum_{j=1}^{k}\left[m_j + \frac{\epsilon}{kV(R_j)}\right]V(R_j) = s(\mathbf{P}) + \epsilon.$$

Now Theorem 1.1.3 implies that $s(\mathbf{P})$ is the infimum of the set of Riemann sums of f over $\mathbf{P}$.

7.1.6. Let $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$. Let

$$M_j = \sup\left\{f(x,y) \,\middle|\, \mathbf{X} \in R_j\right\} \quad \text{and} \quad m_j = \inf\left\{f(x,y) \,\middle|\, \mathbf{X} \in R_j\right\}.$$

Let j be arbitrary in $\{1, 2, \dots, k\}$. Since R_j contains a point $(\widehat{x}_j, \widehat{y}_j)$ with $\widehat{x}_j$ and $\widehat{y}_j$ irrational, $M_j = 3$. Hence, $\overline{\int_R} f(x,y)\,d(x,y) = 3(b-a)(d-c)$. Since R_j contains a point $(\widetilde{x}_j, \widetilde{y}_j)$ with $\widetilde{x}_j$ and $\widetilde{y}_j$ rational, $m_j = 0$. Hence, $\underline{\int_R} f(x,y)\,d(x,y) = 0$.

7.1.7. First suppose that P_1' is obtained by adding one point to P_1, and $P_j' = P_j$, $2 \le j \le n$. If P_i is defined by

$$P_i : a_i = a_{i0} < a_{i1} < \cdots < a_{im_i} = b_i, \quad 1 \le i \le n,$$

then a typical subrectangle of $\mathbf{P}$ is of the form

$$R_{j_1 j_2 \cdots j_n} = [a_{1,j_1-1}, a_{1j_1}] \times [a_{2,j_2-1}, a_{2j_2}] \times \cdots \times [a_{n,j_n-1}, a_{nj_n}].$$

Let c be the additional point introduced into P_1 to obtain P_1', and suppose that $a_{1,k-1} < c < a_{1k}$. If $j_1 \ne k$ then $R_{j_1 j_2 \cdots j_n}$ is common to $\mathbf{P}$ and $\mathbf{P}'$, so the terms associated with it in $s(\mathbf{P})$ and $s(\mathbf{P}')$ cancel in the difference $s(\mathbf{P}') - s(\mathbf{P})$. To analyze the terms that do not cancel, define

$$R^{(1)}_{k,j_2\cdots j_n} = [a_{1,k-1}, c] \times [a_{2,j_2-1}, a_{2j_2}] \times \cdots \times [a_{n,j_n-1}, a_{nj_n}],$$
$$R^{(2)}_{kj_2\cdots j_n} = [c, a_{1k}] \times [a_{2,j_2-1}, a_{2j_2}] \times \cdots \times [a_{n,j_n-1}, a_{nj_n}],$$

$$m_{kj_2\cdots j_n} = \inf\left\{f(\mathbf{X}) \,\middle|\, \mathbf{X} \in R_{kj_2\cdots j_n}\right\} \tag{A}$$

and

$$m^{(i)}_{kj_2\cdots j_n} = \inf\left\{f(\mathbf{X}) \,\middle|\, \mathbf{X} \in R^{(i)}_{kj_2\cdots j_n}\right\}, \quad i = 1, 2; \tag{B}$$

Then $s(\mathbf{P}') - s(\mathbf{P})$ is the sum of terms of the form

$$\begin{aligned}[m^{(1)}_{kj_2\cdots j_n}(c - a_{1,k-1}) + m^{(2)}_{kj_2\cdots j_n}(a_{1k} - c) - m_{kj_2\cdots j_n}(a_{1k} - a_{1,k-1})]\\ \times(a_{2j_2} - a_{2,j_2-1})\cdots(a_{nj_n} - a_{n,j_n-1}).\end{aligned} \tag{C}$$

The terms within the brackets can be rewritten as

$$(m^{(1)}_{kj_2\cdots j_n} - m_{kj_2\cdots j_n})(c - a_{1,k-1}) + (m^{(2)}_{kj_2\cdots j_n} - m_{kj_2\cdots j_n})(a_{1k} - c), \tag{D}$$

which is nonnegative, because of (A) and (B). Therefore, (E) $s(\mathbf{P}) \le s(\mathbf{P}')$. Moreover, the quantity in (D) is not greater than $2M(a_{1k} - a_{1,k-1})$, so (C) implies that the general surviving term in $s(\mathbf{P}') - s(\mathbf{P})$ is not greater than

$$2M\|\mathbf{P}\|(a_{2j_2} - a_{2,j_2-1})\cdots(a_{nj_n} - a_{n,j_n-1}).$$

The sum of these terms as $j_2, \dots, j_n$ assume all possible values ($1 \le j_i \le m_i$, $i = 2, 3, \dots, n$) is

$$2M\|\mathbf{P}\|(b_2 - a_2)\cdots(b_n - a_n) = \frac{2M\|\mathbf{P}\|V(R)}{b_1 - a_1}.$$

This implies that

$$s(\mathbf{P}') \le s(\mathbf{P}) + \frac{2M\|\mathbf{P}\|V(R)}{b_1 - a_1}.$$

This and (E) imply (17) for $r_1 = 1$ and $r_2 = \cdots = r_n = 0$.

Similarly, if $r_i = 1$ for some i in $\{1, \dots, n\}$ and $r_j = 0$ if $j \ne i$, then

$$s(\mathbf{P}') \le s(\mathbf{P}) + \frac{2M\|\mathbf{P}\|V(R)}{b_i - a_i}.$$

To obtain (17) in the general case, repeat this argument $r_1 + r_2 + \cdots + r_n$ times, as in the proof of Lemma 3.2.1.

7.1.8. Suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are partitions of R and $\mathbf{P}'$ is a refinement of both. From Lemma 7.1.5, $s(\mathbf{P}_1) \le s(\mathbf{P}')$ and $S(\mathbf{P}') \le S(\mathbf{P}_2)$. Since $s(\mathbf{P}') \le S(\mathbf{P}')$ this implies that $s(\mathbf{P}_1) \le S(\mathbf{P}_2)$. Thus, every lower sum is a lower bound for the set of all upper sums. Since $\overline{\int_R} f(\mathbf{X})\,d\mathbf{X}$ is the infimum of this set, $s(\mathbf{P}_1) \le \overline{\int_R} f(\mathbf{X})\,d\mathbf{X}$ for every partition $\mathbf{P}_1$ of R. This means that $\overline{\int_R} f(\mathbf{X})\,d\mathbf{X}$ is an upper bound for the set of all lower sums. Since $\underline{\int_R} f(\mathbf{X})\,d\mathbf{X}$ is the supremum of this set, this implies that $\underline{\int_R} f(\mathbf{X})\,d\mathbf{X} \le \overline{\int_R} f(\mathbf{X})\,d\mathbf{X}$.

7.1.9. Suppose that $\mathbf{P}$ is a partition of R and σ is a Riemann sum of f over $\mathbf{P}$. Let $\epsilon > 0$. From Definition 7.1.2, there is a $\delta > 0$ such that

$$\left|\sigma - \int_R f(\mathbf{X})\,d\mathbf{X}\right| < \frac{\epsilon}{3} \quad \text{if} \quad \|\mathbf{P}\| < \delta. \tag{A}$$

From the triangle inequality,

$$\begin{aligned}\left|\overline{\int_R} f(\mathbf{X})\,d\mathbf{X} - \int_R f(\mathbf{X})\,d\mathbf{X}\right| \le {}& \left|\overline{\int_R} f(\mathbf{X})\,d\mathbf{X} - S(\mathbf{P})\right| + |S(\mathbf{P}) - \sigma| \\ &+ \left|\sigma - \int_R f(\mathbf{X})\,d\mathbf{X}\right|.\end{aligned} \tag{B}$$

From Definition 7.1.4, there is a partition $\mathbf{P}_0$ of R such that

$$\overline{\int_R} f(\mathbf{X})\,d\mathbf{X} \le S(\mathbf{P}_0) < \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} + \frac{\epsilon}{3}. \tag{C}$$

Now suppose that $\|\mathbf{P}\| < \delta$ and $\mathbf{P}$ is a refinement of $\mathbf{P}_0$. Since $S(\mathbf{P}) \le S(\mathbf{P}_0)$ by Lemma 7.1.6, (C) implies that

$$\overline{\int_R} f(\mathbf{X})\,d\mathbf{X} \le S(\mathbf{P}) < \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} + \frac{\epsilon}{3}$$

so

$$\left| S(\mathbf{P}) - \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} \right| < \frac{\epsilon}{3} \tag{D}$$

in addition to (A). Now (A), (B), and (D) imply that

$$\left| \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} - \int_R f(\mathbf{X})\,d\mathbf{X} \right| < \frac{2\epsilon}{3} + |S(\mathbf{P}) - \sigma| \tag{E}$$

for every Riemann sum σ of f over $\mathbf{P}$. Since $S(\mathbf{P})$ is the supremum of these Riemann sums (Theorem 7.1.5), we may choose σ so that $|S(\mathbf{P}) - \sigma| < \dfrac{\epsilon}{3}$. Now (E) implies that

$$\left| \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} - \int_R f(\mathbf{X})\,d\mathbf{X} \right| < \epsilon.$$

Since ϵ is an arbitrary positive number, this implies that

$$\overline{\int_R} f(\mathbf{X})\,d\mathbf{X} = \int_R f(\mathbf{X})\,d\mathbf{X}.$$

From the triangle inequality,

$$\begin{aligned} \left| \underline{\int_R} f(\mathbf{X})\,d\mathbf{X} - \int_R f(\mathbf{X})\,d\mathbf{X} \right| \le {} & \left| \underline{\int_R} f(\mathbf{X})\,d\mathbf{X} - s(\mathbf{P}) \right| + |s(\mathbf{P}) - \sigma| \\ & + \left| \sigma - \int_R f(\mathbf{X})\,d\mathbf{X} \right|. \end{aligned} \tag{F}$$

From Definition 7.1.4, there is a partition $\mathbf{P}_1$ of R such that

$$\underline{\int_R} f(\mathbf{X})\,d\mathbf{X} \ge s(\mathbf{P}_1) > \underline{\int_R} f(\mathbf{X})\,d\mathbf{X} - \frac{\epsilon}{3}. \tag{G}$$

Now suppose that $\|\mathbf{P}\| < \delta$ and $\mathbf{P}$ is a refinement of $\mathbf{P}_1$. Since $s(\mathbf{P}) \ge s(\mathbf{P}_1)$ by Lemma 7.1.6, (G) implies that

$$\underline{\int_R} f(\mathbf{X})\,d\mathbf{X} \ge s(\mathbf{P}) > \underline{\int_R} f(\mathbf{X})\,d\mathbf{X} + \frac{\epsilon}{3}$$

so

$$\left| s(\mathbf{P}) - \underline{\int_R} f(\mathbf{X})\, d\mathbf{X} \right| < \frac{\epsilon}{3} \tag{H}$$

in addition to (A). Now (A), (F), and (H) imply that

$$\left| \underline{\int_R} f(\mathbf{X})\, d\mathbf{X} - \int_R f(\mathbf{X})\, d\mathbf{X} \right| < \frac{2\epsilon}{3} + |s(\mathbf{P}) - \sigma| \tag{I}$$

for every Riemann sum σ of f over $\mathbf{P}$. Since $s(\mathbf{P})$ is the infimum of these Riemann sums (Theorem 7.1.5), we may choose σ so that $|s(\mathbf{P}) - \sigma| < \frac{\epsilon}{3}$. Now (I) implies that

$$\left| \underline{\int_R} f(\mathbf{X})\, d\mathbf{X} - \int_R f(\mathbf{X})\, d\mathbf{X} \right| < \epsilon.$$

Since ϵ is an arbitrary positive number, this implies that

$$\underline{\int_R} f(\mathbf{X})\, d\mathbf{X} = \int_R f(\mathbf{X})\, d\mathbf{X}.$$

7.1.10. The inequalities $\overline{\int_R} f(\mathbf{X})\, d\mathbf{X} \le S(\mathbf{P})$ and $\underline{\int_R} f(\mathbf{X})\, d\mathbf{X} \ge s(\mathbf{P})$ follow directly from Definition 7.1.4. Now suppose that $|f(\mathbf{X})| \le K$ if $\mathbf{X} \in R$ and $\epsilon > 0$. Let $R = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$.
From Definition 7.1.4, there is a partition $\mathbf{P}_0 = P_1^{(0)} \times P_2^{(0)} \times \cdots \times P_n^{(0)}$ of R such that (A) $S(\mathbf{P}_0) < \overline{\int_R} f(\mathbf{X})\, d\mathbf{X} + \frac{\epsilon}{2}$. Assume that $\mathbf{P}_j^{(0)}$ has $r_j + 2$ partition points (including a_j and b_j), $1 \le j \le n$. If $\mathbf{P} = P_1 \times P_2 \times \cdots \times P_n$ is any partition of R, let $\mathbf{P}' = P_1' \times P_2' \times \cdots \times P_n'$, where P_j' is the refinement of P_j including the partition points of $P_j^{(0)}$, $1 \le j \le n$. Then Lemma 7.1.6 implies that (B) $S(\mathbf{P}') \le S(\mathbf{P}_0)$ and

$$S(\mathbf{P}') \ge S(\mathbf{P}) - 2KV(R) \left(\sum_{j=1}^{n} \frac{r_j}{b_j - a_j} \right) \|\mathbf{P}\|. \tag{C}$$

Now (A), (B), and (C) imply that

$$\begin{aligned} S(\mathbf{P}) &\le S(\mathbf{P}') + 2KV(R) \left(\sum_{j=1}^{n} \frac{r_j}{b_j - a_j} \right) \|\mathbf{P}\| \\ &\le S(\mathbf{P}_0) + 2KV(R) \left(\sum_{j=1}^{n} \frac{r_j}{b_j - a_j} \right) \|\mathbf{P}\| \\ &< \overline{\int_R} f(\mathbf{X})\, d\mathbf{X} + \frac{\epsilon}{2} + 2KV(R) \left(\sum_{j=1}^{n} \frac{r_j}{b_j - a_j} \right) \|\mathbf{P}\| < \overline{\int_R} f(\mathbf{X})\, d\mathbf{X} + \epsilon \end{aligned}$$

if $\|\mathbf{P}\| < \delta$, where $2KV(R)\left(\sum_{j=1}^{n} \frac{r_j}{b_j - a_j}\right)\delta < \frac{\epsilon}{2}$.

From Definition 7.1.4, there is a partition $\mathbf{P}_1 = P_1^{(1)} \times P_2^{(1)} \times \cdots \times P_n^{(1)}$ of R such that (D) $s(\mathbf{P}_1) > \overline{\int_R} f(\mathbf{X})\, d\mathbf{X} - \frac{\epsilon}{2}$. Assume that $\mathbf{P}_j^{(1)}$ has $s_j + 2$ partition points (including a_j and b_j), $1 \le j \le n$. If $\mathbf{P} = P_1 \times P_2 \times \cdots \times P_n$ is any partition of R, let $\mathbf{P}'' = P_1'' \times P_2'' \times \cdots \times P_n''$, where P_j'' is the refinement of P_j including the partition points of $P_j^{(1)}$, $1 \le j \le n$. Then Lemma 7.1.6 implies that (E) $s(\mathbf{P}'') \ge s(\mathbf{P}_1)$ and

$$s(\mathbf{P}'') \le s(\mathbf{P}) + 2KV(R)\left(\sum_{j=1}^{n} \frac{s_j}{b_j - a_j}\right)\|\mathbf{P}\|. \tag{F}$$

Now (D), (E), and (F) imply that

$$\begin{aligned} s(\mathbf{P}) &\ge s(\mathbf{P}'') - 2KV(R)\left(\sum_{j=1}^{n} \frac{s_j}{b_j - a_j}\right)\|\mathbf{P}\| \\ &\ge s(\mathbf{P}_1) - 2KV(R)\left(\sum_{j=1}^{n} \frac{s_j}{b_j - a_j}\right)\|\mathbf{P}\| \\ &> \underline{\int_R} f(\mathbf{X})\, d\mathbf{X} - \frac{\epsilon}{2} - 2KV(R)\left(\sum_{j=1}^{n} \frac{s_j}{b_j - a_j}\right)\|\mathbf{P}\| > \underline{\int_R} f(\mathbf{X})\, d\mathbf{X} - \epsilon \end{aligned}$$

if $\|\mathbf{P}\| < \delta$, where $2KV(R)\left(\sum_{j=1}^{n} \frac{s_j}{b_j - a_j}\right)\delta < \frac{\epsilon}{2}$.

7.1.11. If $\epsilon > 0$, there is a $\delta > 0$ such that

$$\underline{\int_R} f(\mathbf{X})\, d\mathbf{X} - \epsilon < s(\mathbf{P}) \le S(\mathbf{P}) < \overline{\int_R} f(\mathbf{X})\, d\mathbf{X} + \epsilon \quad \text{if} \quad \|\mathbf{P}\| < \delta$$

(Lemma 7.1.9). By assumption, this is equivalent to

$$L - \epsilon < s(\mathbf{P}) \le S(\mathbf{P}) < L + \epsilon \quad \text{if} \quad \|\mathbf{P}\| < \delta. \tag{A}$$

If σ is a Riemann sum of f over $\{\mathbf{P}\}$ then $s(\mathbf{P}) \le \sigma \le S(\mathbf{P})$, so (A) implies that $L - \epsilon < \sigma < L + \epsilon$ if $\|\mathbf{P}\| < \delta$. Now Definition 7.1.2 implies that $\int_R f(\mathbf{X})\, d\mathbf{X} = L$.

7.1.12. Suppose that $\int_R f(\mathbf{X})\, d\mathbf{X} = L$. If $\epsilon > 0$, there is a $\delta > 0$ such that $L - \epsilon/3 < \sigma < L + \epsilon/3$ if σ is any Riemann sum of f over a partition $\mathbf{P}$ with $\|\mathbf{P}\| < \delta$. Since $s(\mathbf{P})$ and S($\mathbf{P}$) are respectively the infimum and supremum of all Riemann sums of f over $\mathbf{P}$, (Theorem 7.1.5) it folllows that $L-\epsilon/3 \le s(\mathbf{P}) \le S(\mathbf{P}) \le L+\epsilon/3$ if $\|\mathbf{P}\| < \delta$. Therefore, $|S(\mathbf{P})-s(\mathbf{P})| < \epsilon$ if $\|\mathbf{P}\| < \delta$.

Now suppose that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|S(\mathbf{P})-s(\mathbf{P})| < \epsilon$ if $\|\mathbf{P}\| < \delta$. Since $s(\mathbf{P}) \le \underline{\int_R} f(\mathbf{X})\,d\mathbf{X} \le \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} \le S(\mathbf{P})$ for all $\mathbf{P}$, this implies that $0 \le \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} - \underline{\int_R} f(\mathbf{X})\,d\mathbf{X} < \epsilon$. Since ϵ is an arbitrary positive number, this implies that $\overline{\int_R} f(\mathbf{X})\,d\mathbf{X} = \underline{\int_R} f(\mathbf{X})\,d\mathbf{X}$. Therefore, $\int_R f(\mathbf{X})\,d\mathbf{X}$ exists, by Theorem 7.1.7.

7.1.14. **(a)** If $\epsilon > 0$, there are rectangles $T_1, T_2, \dots, T_r$ and $T_1', T_2', \dots, T_s'$ such that $S_1 \subset \cup_{i=1}^r T_i$, $S_2 \subset \cup_{j=1}^s T_j'$, $\sum_{i=1}^r V(T_r) < \epsilon/2$, and $\sum_{j=1}^s V(T_j') < \epsilon/2$. Then $\{T_1, \dots, T_m, T_1', \dots, T_r'\}$ covers $S_1 \cup S_2$ with total content $< \epsilon$.

(b) If $S_1 \subset \cup_{i=1}^r T_r$ and $S_2 \subset S_1$, then $S_2 \subset \cup_{i=1}^r T_r$.

(c) Since $\cup_{i=1}^r T_i$ is closed, $\overline{S} \subset \cup_{i=1}^r T_i$ if $S \subset \cup_{i=1}^r T_i$.

7.1.15. Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $T = [a_1-\epsilon, b_1+\epsilon] \times \cdots \times [a_n-\epsilon, b_n+\epsilon]$, where $\epsilon > 0$. Then $R \subset T$ and, since $a_i = b_i$ for some i, $V(T) \le 2\epsilon L^{n-1}$, where $L = \max\left\{(b_j - a_j) \,\middle|\, 1 \le j \le n\right\}$.

7.1.16. Let R be a rectangle containing S. Suppose that $\epsilon > 0$. Since f is uniformly continous on S (Theorem 7.1.5.2.15), there is a partition P of R such that $|f(\mathbf{X})-f(\mathbf{Y})| < \epsilon$ if $\mathbf{X}$ and $\mathbf{Y}$ are in S and in the same subrectangle of P. If $R_1, R_2, \dots, R_m$ are the subrectangles of P such that $S \cap R_i \ne \emptyset$ and $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_m$ are points in $R_1, R_2, \dots, R_m$, then the surface is contained in the set $\cup_{i=1}^m \left\{(\mathbf{X}, z) \,\middle|\, \mathbf{X} \in R_i \text{ and } |z - f(\mathbf{X}_i)| < \epsilon\right\}$. The content of this set (in R^{n+1}) is $< K\epsilon$, where K is independent of ϵ.

7.1.17. **(a)** Since T does not have zero content, there is an $\epsilon_0 > 0$ such that any finite collection of rectangles covering T has total content $\ge \epsilon_0$. Let R be a rectangle containing S, and let $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ be a partition of R. Let $U = \left\{j \,\middle|\, R_j^0 \cap T \ne \emptyset\right\}$. Then $\sum_{j\in U} V(R_j) \ge \epsilon_0$, since the total content of $\cup_{j=1}^k \partial R_j$ is zero (Exercise 7.1.15). If $j \in U$, then R_j^0 contains points in T and points not in S, so $\sup\left\{f_S(\mathbf{X}) \,\middle|\, \mathbf{X} \in R_j\right\} - \inf\left\{f_S(\mathbf{X}) \,\middle|\, \mathbf{X} \in R_j\right\} \ge \rho$; therefore, if $S(\mathbf{P})$ and $s(\mathbf{P})$ are upper and lower sums of f_S over R, then $S(\mathbf{P}) - s(\mathbf{P}) \ge \rho\epsilon_0$. Hence, f_S is not integrable on R (Theorem 7.1.12), so f is not integrable on S (Definition 7.1.17).

(b) If $f \equiv 1$, then $f(\mathbf{X}) \ge 1$ on $T = S \cap \partial S$. Hence, $\int_S d\mathbf{X}$, from **(a)**.

7.1.18. **(a)** Suppose that $|h(\mathbf{X})| \le K$ and $S_0 = \left\{\mathbf{X} \,\middle|\, \mathbf{X} \in S \quad\text{and}\quad f(\mathbf{X}) \ne 0\right\}$. Let $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be a rectangle containing S. If $\epsilon > 0$, let $T_1, T_2, \dots, T_m$ be subrectangles of R such that $S_0 \subset \cup_{j=1}^m T_j$ and (A) $\sum_{j=1}^m V(T_j) < \epsilon$. ($S_0$ has zero content, from Exercise 7.1.14**(b)**. If $T_j = [\alpha_{1j}, \beta_{1j}] \times \cdots \times [\alpha_{nj}, \beta_{nj}]$, let $\mathbf{P} = P_1 \times P_2 \times \cdots \times P_k$, where $\mathbf{P}_i$ is the partition of $[a_i, b_i]$ with partition points

$$a_i, b_i, \alpha_{i1}, \beta_{i1}, \alpha_{i2}, \beta_{i2}, \dots, \alpha_{im}, \beta_{im}.$$

(These are not in order.) We may suppose that $S_0 \cap R^0 \subset \cup_{j=1}^m T_j^0$, since if this were not so, we could simply enlarge $T_1, T_2, \dots, T_m$ slightly to make it so, while maintaining (A). This guarantees that the union of the subrectangles of $\mathbf{P}$ on which h_S assumes nonzero values is contained in $\cup_{j=1}^m T_j$; hence, (B) $|S(\mathbf{P})| \le K\epsilon$ and (C) $|s(\mathbf{P})| \le K\epsilon$ if $S(\mathbf{P})$ and $s(\mathbf{P})$ are upper and lower sums for h_S over R. This implies that $S(\mathbf{P}) - s(\mathbf{P}) \le 2k\epsilon$, so $\int_R h_S(\mathbf{X})\,d\mathbf{X} = \int_S h(\mathbf{X})\,d\mathbf{X}$ exists (Theorem 7.1.12. If σ is a Riemann sum for h_S over any refinement $\mathbf{P}'$ of $\mathbf{P}$, then $|\sigma| \le K\epsilon$. This implies that $\int_R h_S(\mathbf{X})\,d\mathbf{X} = 0$, so $\int_S h(\mathbf{X})\,d\mathbf{X} = 0$.

(b) From **(a)** with $h = g - f$, $\int_S (g - f)(\mathbf{X})\,d\mathbf{X} = 0$. Theorem 1.11 implies that $\int_S g(\mathbf{X})\,d\mathbf{X} = \int_S f(\mathbf{X})\,d\mathbf{X} + \int_S (g - f)(\mathbf{X})\,d\mathbf{X}$.

7.1.19. Let $\epsilon > 0$. Since ∂S_0 has zero content, there are rectangles $T_1, T_2, \dots, T_m$ such that $\sum_{j=1}^m V(T_j) < \epsilon$ and $\partial S_0 \subset \cup_{j=1}^m T_j$. Let $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be a rectangle such that $S \cup (\cup_{j=1}^m T_j) \subset R$. Let $S(\mathbf{P})$ and $s(\mathbf{P})$ be upper and lower sums for f_S over partitions of R, and let $\overline{S}(\mathbf{P})$ and $\overline{s}(\mathbf{P})$ be similarly defined for f_{S_0}.

If $T_j = [\alpha_{1j}, \beta_{1j}] \times \cdots \times [\alpha_{nj}, \beta_{nj}]$, let $\mathbf{P}_0 = P_1 \times P_2 \times \cdots \times P_k$, where $\mathbf{P}_i$ is the partition of $[a_i, b_i]$ with partition points

$$a_i, b_i, \alpha_{i1}, \beta_{i1}, \alpha_{i2}, \beta_{i2}, \dots, \alpha_{im}, \beta_{im}.$$

(These are not in order.) From Theorem 7.1.12 and Lemma 7.1.6, there is a partition $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ of R such that $S(P) - s(P) < \epsilon$ and P is a refinement of $\mathbf{P}_0$. Now let $T_0 = \left\{ j \mid R_j \cap \partial S_0 \ne \emptyset \right\}$ and suppose that $|f(\mathbf{X})| \le M$, $\mathbf{X} \in S$. Then

$$\overline{S}(P) - \overline{s}(P) \le S(P) - s(P) + 2M \sum_{j \in T_0} V(R_j) < (2M + 1)\epsilon.$$

7.1.20. First suppose that $S = R$, a rectangle. Any Riemann sum of $f + g$ over a partition $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ of R can be written as

$$\begin{aligned}
\sigma_{f+g} &= \sum_{j=1}^k [f(\mathbf{X}_j) + g(\mathbf{X}_j)]V(R_j) \\
&= \sum_{j=1}^k f(\mathbf{X}_j)V(R_j) + \sum_{j=1}^k g(\mathbf{X}_j)V(R_j) \\
&= \sigma_f + \sigma_g,
\end{aligned}$$

where σ_f and σ_g are Riemann sums for f and g. Definition 7.1.2 implies that if $\epsilon > 0$, there are positive numbers δ_1 and δ_2 such that

$$\left|\sigma_f - \int_R f(\mathbf{X})\, d\mathbf{X}\right| < \frac{\epsilon}{2} \quad \text{if} \quad \|\mathbf{P}\| < \delta_1$$

and

$$\left|\sigma_g - \int_R g(x)\, d\mathbf{X}\right| < \frac{\epsilon}{2} \quad \text{if} \quad \|\mathbf{P}\| < \delta_2.$$

If $\|\mathbf{P}\| < \delta = \min(\delta_1, \delta_2)$ then

$$\begin{aligned}\left|\sigma_{f+g} - \int_R f(\mathbf{X})\, d\mathbf{X} - \int_R g(x)\, d\mathbf{X}\right| &= \left|\left(\sigma_f - \int_R f(\mathbf{X})\, d\mathbf{X}\right) + \left(\sigma_g - \int_R g(x)\, d\mathbf{X}\right)\right| \\ &\le \left|\sigma_f - \int_R f(\mathbf{X})\, d\mathbf{X}\right| + \left|\sigma_g - \int_R g(x)\, d\mathbf{X}\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

so Definition 7.1.2 implies the conclusion.

To obtain the conclusion for a general bounded set S, let R be a rectangle containing S and apply this result with f and g replaced by f_S and g_S.

7.1.21. First, consider the case where $S = R$ (rectangle). The conclusion is trivial if $c = 0$. Suppose that $c \ne 0$ and $\epsilon > 0$. Any Riemann sum of cf over a partition $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ of R can be written as $\widehat{\sigma} = \sum_{j=1}^{k} cf(\mathbf{X}_j)V(R_j) = c\sum_{j=1}^{k} f(\mathbf{X}_j)V(R_j) = c\sigma$, where σ is a Riemann sum of f over $\mathbf{P}$. Since f is integrable on R, Definition 7.1.2 implies that there is a $\delta > 0$ such that $\left|\sigma - \int_R f(\mathbf{X})\, d\mathbf{X}\right| < \frac{\epsilon}{|c|}$ if $\|\mathbf{P}\| < \delta$. Therefore, $\left|\widehat{\sigma} - \int_R cf(\mathbf{X})\, d\mathbf{X}\right| < \epsilon$ if $\widehat{\sigma}$ $\|\mathbf{P}\| < \delta$, so cf is integrable over R, again by Definition 7.1.2.

To obtain the conclusion for a general bounded set S, let R be a rectangle containing S and apply this result with f replaced by f_S.

7.1.22. First, consider the case where $S = R$ (rectangle). Since $g(\mathbf{X}) - f(\mathbf{X}) \ge 0$, every lower sum of $g - f$ over any partition of R is nonnegative. Therefore,

$$\underline{\int_R} [g(\mathbf{X}) - f(\mathbf{X})]\ d\mathbf{X} \ge 0.$$

Hence,

$$\begin{aligned}\int_R g(\mathbf{X})\, d\mathbf{X} - \int_R f(\mathbf{X})\, d\mathbf{X} &= \int_R [g(\mathbf{X}) - f(\mathbf{X})]\, d\mathbf{X} \\ &= \underline{\int_R} [g(\mathbf{X}) - f(\mathbf{X})]\ d\mathbf{X} \ge 0,\end{aligned} \tag{A}$$

so $\int_R f(\mathbf{X})\, d\mathbf{X} \le \int_R g(\mathbf{X})\, d\mathbf{X}$. (The first equality in (A) follows from Theorems 7.1.23 and 7.1.24; the second, from Theorem 7.1.8).

To obtain the conclusion for a general bounded set S, let R be a rectangle containing S and apply this result with f and g replaced by f_S and g_S.

7.1.23. First, consider the case where $S = R$ (rectangle). Let $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ be a partition of R and define

$$\begin{aligned} M_j &= \sup\left\{f(\mathbf{X}) \,\middle|\, \mathbf{X} \in R_j\right\}, \\ m_j &= \inf\left\{f(\mathbf{X}) \,\middle|\, \mathbf{X} \in R_j\right\}, \\ \overline{M}_j &= \sup\left\{|f(\mathbf{X})| \,\middle|\, \mathbf{X} \in R_j\right\}, \\ \overline{m}_j &= \inf\left\{|f(\mathbf{X})| \,\middle|\, \mathbf{X} \in R_j\right\}. \end{aligned}$$

Then

$$\begin{aligned} \overline{M}_j - \overline{m}_j &= \sup\left\{|f(\mathbf{X})| - |f(\mathbf{X}')| \,\middle|\, \mathbf{X}, \mathbf{X}' \in R_j\right\} \\ &\le \sup\left\{|f(\mathbf{X}) - f(\mathbf{X}')| \,\middle|\, \mathbf{X}, \mathbf{X}' \in R_j\right\} \\ &= M_j - m_j. \end{aligned} \tag{A}$$

Therefore, $\overline{S}(\mathbf{P}) - \overline{s}(\mathbf{P}) \le S(\mathbf{P}) - s(\mathbf{P})$, where the upper and lower sums on the left are associated with $|f|$ and those on the right are associated with f. Now suppose that $\epsilon > 0$. Since f is integrable on R, Theorem 7.1.12 implies that there is a partition $\mathbf{P}$ of R such that $S(\mathbf{P}) - s(\mathbf{P}) < \epsilon$. This and (A) imply that $\overline{S}(\mathbf{P}) - \overline{s}(\mathbf{P}) < \epsilon$. Therefore, $|f|$ is integrable on R, again by Theorem 7.1.12.

Since $f(\mathbf{X}) \le |f(\mathbf{X})|$ and $-f(\mathbf{X}) \le |f(\mathbf{X})|$, $\mathbf{X} \in R$, Theorems 7.1.2 and 7.1.4 imply that $\int_R f(\mathbf{X})\,d\mathbf{X} \le \int_R |f(\mathbf{X})|\,d\mathbf{X}$ and $-\int_R f(\mathbf{X})\,d\mathbf{X} \le \int_R |f(\mathbf{X})|\,d\mathbf{X}$, so $\left|\int_S f(\mathbf{X})\,d\mathbf{X}\right| \le \int_S |f(\mathbf{X})|\,d\mathbf{X}$.

To obtain the conclusion for a general bounded set S, let R be a rectangle containing S and apply this result with f replaced by f_S.

7.1.24. First, consider the case where $S = R$ (rectangle) and f and g are nonnegative. The subscripts f, g, and fg in the following argument identify the functions with which the various quantities are associated. We assume that neither f nor g is identically zero on R, since the conclusion is obvious if one of them is.

If $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ is a partition of R then

$$S_{fg}(\mathbf{P}) - s_{fg}(p) = \sum_{j=1}^{k} (M_{fg,j} - m_{fg,j}) V(R_j). \tag{A}$$

Since f and g are nonnegative, $M_{fg,j} \le M_{f,j} M_{g,j}$ and $m_{fg,j} \ge m_{f,j} m_{g,j}$. Hence,

$$\begin{aligned} M_{fg,j} - m_{fg,j} &\le M_{f,j} M_{g,j} - m_{f,j} m_{g,j} \\ &= (M_{f,j} - m_{f,j}) M_{g,j} + m_{f,j}(M_{g,j} - m_{g,j}) \\ &\le M_g(M_{f,j} - m_{f,j}) + M_f(M_{g,j} - m_{g,j}), \end{aligned}$$

where M_f and M_g are upper bounds for f and g on R. From (A) and the last inequality,

$$S_{fg}(\mathbf{P}) - s_{fg}(\mathbf{P}) \le M_g[S_f(\mathbf{P}) - s_f(\mathbf{P})] + M_f[S_g(\mathbf{P}) - s_g(\mathbf{P})]. \tag{B}$$

Now suppose that $\epsilon > 0$. Theorem 7.1.12 implies that there are partitions $\mathbf{P}_1$ and $\mathbf{P}_2$ of R such that

$$S_f(\mathbf{P}_1) - s_f(\mathbf{P}_1) < \frac{\epsilon}{2M_g} \quad \text{and} \quad S_g(\mathbf{P}_2) - s_g(\mathbf{P}_2) < \frac{\epsilon}{2M_f}. \tag{C}$$

From Lemma 7.1.6, the inequalities in (C) also hold for any partition $\mathbf{P}$ that is a refinement of $\mathbf{P}_1$ and $\mathbf{P}_2$; hence, (B) yields

$$S_{fg}(\mathbf{P}) - s_{fg}(\mathbf{P}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for any such partition. Therefore, fg is integrable on R, by Theorem 7.1.12.

If $f(\mathbf{X}) \ge m_1$ and $g(\mathbf{X}) \ge m_2$ ($\mathbf{X} \in R$), write $fg = (f - m_1)(g - m_2) + m_2 f + m_1 g - m_1 m_2$. The first product on the right is integrable by the proof given above. To complete the proof, use Theorems 7.1.23 and 7.1.24.

To obtain the conclusion for a general bounded set S, let R be a rectangle containing S and apply this result with f and g replaced by f_S and g_S.

7.1.25. From Theorem 7.1.13, u is integrable on R. Therefore, Theorem 7.1.27 implies that $\int_R u(\mathbf{X})v(\mathbf{X})\,d\mathbf{X}$ exists. If $m = \min\left\{u(\mathbf{X}) \,\middle|\, \mathbf{X} \in R\right\}$ and $M = \max\left\{u(\mathbf{X}) \,\middle|\, \mathbf{X} \in R\right\}$, then $m \le u(\mathbf{X}) \le M$ and, since $v(\mathbf{X}) \ge 0$, $mv(\mathbf{X}) \le u(\mathbf{X})v(\mathbf{X}) \le Mv(\mathbf{X})$. Therefore, Theorems 7.1.24 and 7.1.25 imply that

$$m \int_R v(\mathbf{X})\,d\mathbf{X} \le \int_R u(\mathbf{X})v(\mathbf{X})\,d\mathbf{X} \le M \int_R v(\mathbf{X})\,d\mathbf{X}. \tag{A}$$

This implies that

$$\int_R u(\mathbf{X})v(\mathbf{X})\,d\mathbf{X} = u(\mathbf{X}_0) \int_R v(\mathbf{X})\,d\mathbf{X} \tag{B}$$

for any $\mathbf{X}_0$ in R if $\int_R v(\mathbf{X})\,d\mathbf{X} = 0$. If $\int_R v(\mathbf{X})\,d\mathbf{X} \ne 0$, let

$$\overline{u} = \frac{\displaystyle\int_R u(\mathbf{X})v(\mathbf{X})\,d\mathbf{X}}{dst \int_R v(\mathbf{X})\,d\mathbf{X}}$$

Since $\int_R v(\mathbf{X})\,d\mathbf{X} > 0$ in this case, (A) implies that $m \le \overline{u} \le M$, and (Theorem 5.2.13) implies that $\overline{u} = u(\mathbf{X}_0)$ for some $\mathbf{X}_0$ in R. This implies (B).

7.1.26. Let $R' = [a_1', b_1'] \times [a_2', b_2'] \times \cdots \times [a_n', b_n']$. be a subrectangle of $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Suppose that $\epsilon > 0$. From Theorem 7.1.12, there is a partition $\mathbf{P} = P_1 \times P_2 \times \cdot \times P_n = \{R_1, R_2 \dots, R_k\}$ of R such that

$$S(\mathbf{P}) - s(\mathbf{P}) = \sum_{j=1}^{k} (M_j - m_j)V(R_j) < \epsilon. \tag{A}$$

We may assume that a_i' and b_i' are partition points of P_i, $1 \le i \le n$, because if not, they can be inserted to obtain a refinement $\widehat{\mathbf{P}}$ such that $S(\widehat{\mathbf{P}'}) - s(\widehat{\mathbf{P}'}) \le S(\mathbf{P}) - s(\mathbf{P})$ (Lemma 7.1.6).

Therefore, there is a subset U of $\{1, 2, \dots, k\}$ such that $\mathbf{P}' = \{R_j\}_{j\in U}$ is a partition of R'. Since (A) implies that

$$S(\mathbf{P}') - s(\mathbf{P}') = \sum_{j\in U}(M_j - m_j)V(R_j) < \epsilon,$$

Theorem 7.1.12 implies that f is integrable on R'.

7.1.27. **(a)** From Exercise 7.1.26, $\int_R g(\mathbf{X})d\mathbf{X}$ exists if $\int_{\widetilde{R}} g(\mathbf{X})d\mathbf{X}$ does. For the converse, suppose that $\int_R g(\mathbf{X})\,d\mathbf{X}$ exists. Let $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ and $\widetilde{R} = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \cdots \times [\alpha_n, \beta_n]$. Consider only partitions $\widetilde{P} = \widetilde{P}_1 \times \cdots \times \widetilde{P}_n$ of $\widetilde{R}$, where $\widetilde{P}_i$ is a partition of $[\alpha_i, \beta_i]$ that includes a_i and b_i among its partition points. Let P_i be the partition of $[a_i, b_i]$ consisting of the partition points of $\widetilde{P}_i$ in $[a_i, b_i]$. Then $P = P_1 \times P_2 \times \cdots \times P_n$ is a partition of R. Now $\widetilde{P} = \{R_1, \dots, R_m, R_1', \dots, R_k', R_1'', \dots, R_s''\}$, where $P = \{R_1, \dots, R_m\}$, $R_1', \dots, R_k'$ intersect ∂R, but not R^0, and $R_i'' \cap R = \emptyset$ $(1 \le i \le s)$. Now suppose that $|g(\mathbf{X})| \le M$. Since $g(\mathbf{X}) = 0$ if $x \in R_i''$,

$$S_{\widetilde{R}}(\widetilde{P}) - s_{\widetilde{R}}(\widetilde{P}) \le S_R(P) - s_r(P) + 2M\sum_{j=1}^{k} V(R_j'). \tag{A}$$

Since

$$R \cup \left[\cup_{j=1}^{k} R_j'\right] \subset [a_1 - \|\widetilde{P}\|, b_1 + \|\widetilde{P}\|] \times \cdots \times [a_n - \|\widetilde{P}\|, b_1 + \|\widetilde{P}\|],$$

the above stated properties of $R_1', \dots, R_k'$ imply that

$$\sum_{j=1}^{k} V(R_j') \le (b_1 - a_1 + 2\|\widetilde{P}\|)\cdots(b_n - a_n + 2\|\widetilde{P}\| - (b_1 - a_1)\cdots(b_n - a_n). \tag{B}$$

Now suppose that $\epsilon > 0$. Since $\int_R g(\mathbf{X})\,d\mathbf{X}$ exists, there is a $\delta > 0$ such that $S_R(P) - S_R(P) < \epsilon/2$ if $\|P\| < \delta$ (Theorem 7.1.12); in addition, choose δ so that the right side of (B) $< \epsilon/2$ if $\|\widetilde{P}\| < \delta$. Then (A) implies that if $\|\widetilde{P}\| < \delta$, then $S_{\widetilde{R}}(\widetilde{P}) - s_{\widetilde{R}}(\widetilde{P}) < \epsilon$, so $\int_{\widetilde{R}} g(\mathbf{X})d\mathbf{X}$ exists (Theorem 7.1.12).

To see that $\int_{\widetilde{R}} g(\mathbf{X})d\mathbf{X} = \int_R g(\mathbf{X})d\mathbf{X}$, observe that

$$\sigma = \sum_{i=1}^{m} g(x_i)V(R_i) + \sum_{j=1}^{k} 0 \cdot V(R_j') + \sum_{q=1}^{s} 0 \cdot V(R_q''),$$

where $\mathbf{X}_i \in R_i$, can be interpreted as a Riemann sum for both integrals.

(b) Suppose that R_1 and R_2 are rectangles containing S and $\int_{R_1} f_S(\mathbf{X})\,d\mathbf{X}$ exists. Applying **(a)** with $\widetilde{R} = R_1$ and $R = R_1 \cap R_2$ implies that (C) $\int_{R_1\cap R_2} f_S(\mathbf{X})\,d\mathbf{X} = \int_{R_1} f_S(\mathbf{X})\,d\mathbf{X}$. Now applying **(a)** with $\widetilde{R} = R_2$ and $R = R_1\cap R_2$ implies that $\int_{R_2} f_S(\mathbf{X})\,d\mathbf{X} = \int_{R_1\cap R_2} f_S(\mathbf{X})\,d\mathbf{X}$. This and (C) imply the conclusion.

7.1.28. **(a)** From Exercise 7.1.26, f is integrable on each subrectangle $R_1, R_2, \dots, R_k$. If σ_j is a Riemann sum for $\int_{R_j} f(\mathbf{X})\,d\mathbf{X}$ over a partition of R_j, $1 \le j \le k$, then (A) $\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_k$ is a Riemann sum for $\int_R f(\mathbf{X})\,d\mathbf{X}$ over a refinement $\mathbf{P}'$ of $\mathbf{P}$. Because of (A),

$$\int_R f(\mathbf{X})\,d\mathbf{X} - \sum_{j=1}^{k}\int_{R_j} f(\mathbf{X})\,d\mathbf{X} = \left(\int_R f(\mathbf{X})\,d\mathbf{X} - \sigma\right) - \sum_{j=1}^{k}\left(\int_{R_j} f(\mathbf{X})\,d\mathbf{X} - \sigma_j\right) \tag{B}$$

Now suppose that $\epsilon > 0$. By Definition 7.1.2, there is a a $\delta > 0$ such that $\left|\sigma - \int_R f(\mathbf{X})\,d\mathbf{X}\right| < \epsilon$ and $\left|\sigma_j - \int_{R_j} f(\mathbf{X})\,d\mathbf{X}\right| < \epsilon/k$, $1 \le j \le k$, if $\|\mathbf{P}'\| < \delta$. This and (B) imply that

$$\left|\int_R f(\mathbf{X})\,d\mathbf{X} - \sum_{j=1}^{k}\int_{R_j} f(\mathbf{X})\,d\mathbf{X}\right| < 2\epsilon.$$

Since ϵ can be made arbitrarily small, it follows that

$$\int_R f(\mathbf{X})\,d\mathbf{X} - \sum_{j=1}^{k}\int_{R_j} f(\mathbf{X})\,d\mathbf{X} = 0.$$

(b) From **(a)** and Theorem 7.1.28 with $u = f$ and $v = 1$, there is an $\overline{\mathbf{X}}_j \in R_j$ such that $\int_{R_j} f(\mathbf{X})\,d\mathbf{X} = f(\overline{\mathbf{X}}_j)\int_{R_j} d\mathbf{X} = f(\overline{\mathbf{X}}_j)V(R_j)$. The required Riemann sum is $\sigma = \sum_{j=1}^{k} f(\overline{\mathbf{X}}_j)V(R_j) = \int_R f(\mathbf{X})\,d\mathbf{X}$.

7.1.29. Let $P = \{R_1, R_2, \dots, R_k\}$ and (A) $\int_R f(\mathbf{X})\,d\mathbf{X} = \sum_{j=1}^{k} f(\overline{\mathbf{X}}_j)V(R_j)$, where $\overline{\mathbf{X}}_j \in R_j$ (Exercise 7.1.28**(b)**). If $\sigma = \sum_{j=1}^{k} f(\mathbf{X}_j)V(R_j)$ is any Riemann sum over R, then (A) implies that (B) $\left|\sigma - \int_R f(\mathbf{X})\,d\mathbf{X}\right| \le \sum_{j=1}^{k} |f(\mathbf{X}_j) - f(\overline{\mathbf{X}}_j)|V(R_j)$. From Theorem 5.4.5, $f(\mathbf{X}_j) - f(\overline{\mathbf{X}}_j) = \sum_{i=1}^{n} f_{x_i}(\widehat{\mathbf{X}}_j)(x_{ij} - \overline{x}_{ij})$, where $\widehat{\mathbf{X}}_j$ is on the line segment connecting $\mathbf{X}_j$ and $\overline{\mathbf{X}}_j$. From this and Schwarz's inequality, (C) $|f(\mathbf{X}_j) - f(\overline{\mathbf{X}}_j)| \le K|\mathbf{X}_j - \overline{\mathbf{X}}_j|$, where

$$K = \max\left\{\left(\sum_{j=1}^{n} f_{x_j}^2(\mathbf{X})\right)^{1/2} \,\middle|\, \mathbf{X} \in R\right\}.$$

Now (B) and (C) imply that

$$\left|\sigma - \int_R f(\mathbf{X})\,d\mathbf{X}\right| \le K \sum_{j=1}^{k} |\mathbf{X}_j - \overline{\mathbf{X}}_j| V(R_j) \le K\sqrt{n}\|\mathbf{P}\| V(R).$$

Let $M = k\sqrt{n}V(R)$.

7.1.30. **(a)** Let

$$P_1 : a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad P_2 : c = y_0 < y_1 < \cdots < y_s = d$$

be partitions of $[a, b]$ and $[c, d]$, and $\mathbf{P} = P_1 \times P_2$. If we consider f as a function of (x, y) which happens to be independent of y, then $S_f(\mathbf{P}) = (d - c)S_f(P_1)$ and $s_f(\mathbf{P}) = (d - c)s_f(P_1)$. Let $\epsilon > 0$. Since f is integrable over $[a, b]$, there is $\delta > 0$ such that $S_f(P_1) - s_f(P_1) < \epsilon/(d - c)$ if $\|P_1\| < \delta$ (Theorem 3.2.7). Therefore, $S_f(\mathbf{P}) - s_f(\mathbf{P}) < \epsilon$ if $\|\mathbf{P}\| < \delta$, so f is integrable on R (Theorem 7.1.12). Similarly, g is integrable over R.

(c) Since fg is now known to be integrable on R, we may consider Riemann sums of the special form indicated in Exercise 7.1.3 to get the stated equality.

7.1.31. **(a)** Suppose that R is a rectangle containing S. From Exercise 7.1.18**(b)** with $S = R$, $f = f_S$, and $g = f_{S_0}$, $\int_R f_S(\mathbf{X})\,d\mathbf{X} = \int_R f_{S_0}(\mathbf{X})\,d\mathbf{X}$. This implies the conclusion.

(b) Write $S = T_1 \cup T_2 \cup T_3$, where $T_1 = S_1 - S_1 \cap S_2$, $T_2 = S_2 - S_1 \cap S_2$, $T_3 = S_1 \cap S_2$. From **(a)** and Exercise 7.1.14**(b)**, f is integrable on T_1 and T_2; from Exercise 7.1.18**(a)**, $\int_{T_3} f(\mathbf{X})\,d\mathbf{X} = 0$. Now use Theorem 7.1.30.

7.2 ITERATED INTEGRALS AND MULTIPLE INTEGRALS

7.2.1.

(a) $\int_0^2 dy \int_{-1}^1 (x+3y)\,dx = \int_0^2 \left[\left(\frac{x^2}{2} + 3xy\right)\Big|_{x=-1}^{1}\right] dy = \int_0^2 6y\,dy = 3y^2\Big|_0^2 = 12.$

(b)
$$\int_1^2 dx\int_0^1 (x^3+y^4)\,dy = \int_1^2\left[\left(x^3y+\frac{y^5}{5}\right)\Bigg|_{y=0}^1\right]dx$$
$$= \int_1^2\left(x^3+\frac{1}{5}\right)dx = \left(\frac{x^4}{4}+\frac{x}{5}\right)\Bigg|_1^2 = \frac{79}{20}.$$

(c)
$$\int_{\pi/2}^{2\pi} x\,dx\int_1^2 \sin xy\,dy = \int_{\pi/2}^{2\pi} dx\int_1^2 x\sin xy\,dy$$
$$= -\int_{\pi/2}^{2\pi}\left[\cos xy\Big|_{y=1}^2\right]dx$$
$$= \int_{\pi/2}^{2\pi}(\cos x-\cos 2x)\,dx$$
$$= \left(\sin x-\frac{\sin 2x}{2}\right)\Bigg|_{\pi/2}^{2\pi} = -1.$$

(d)
$$\int_0^{\log 2} y\,dy\int_0^1 xe^{x^2y}dx = \int_0^{\log 2} dy\int_0^2 xye^{x^2y}\,dx$$
$$= \frac{1}{2}\int_0^{\log 2}\left[e^{x^2y}\Big|_{x=0}^1\right]dy$$
$$= \frac{1}{2}\int_0^{\log 2}(e^y-1)\,dy$$
$$= \frac{1}{2}(e^y-y)\Big|_0^{\log 2} = \frac{1-\log 2}{2}.$$

7.2.2. Let $P_1: a_1 = x_0 < x_1 < \cdots < x_r = b_1$, $P_2: a_2 = y_0 < y_1 < \cdots < y_s = b_2$, and $P_3: a_3 = z_0 < z_1 < \cdots < z_t = b_3$ be partitions of $[a_1,b_1]$, $[a_2,b_2]$, and $[a_3,b_3]$, and let $\mathbf{P} = P_1\times P_2\times P_3$. Denote $R_{ijk} = [x_{i-1},x_i]\times[y_{j-1},y_j]\times[z_{k-1},z_k]$.

(a) $P_2\times P_3$ is a partition of $I_2\times I_3$. Suppose that

$$(\eta_{jk},\zeta_{jk})\in[y_{j-1},y_j]\times[z_{k-1},z_k],\quad 1\le j\le s,\quad 1\le k\le t, \tag{A}$$

so

$$\sigma = \sum_{j=1}^{s}\sum_{k=1}^{t} G(\eta_{jk},\zeta_{jk})(y_j-y_{j-1})(z_k-z_{k-1}) \tag{B}$$

is a typical Riemann sum of G over $P_2\times P_3$. Since

$$G(\eta_{jk},\zeta_{jk}) = \int_{a_1}^{b_1} f(x,\eta_{jk},\zeta_{jk})\,dx = \sum_{i=1}^{r}\int_{x_{i-1}}^{x} f(x,\eta_{jk},\zeta_{jk})\,dx,$$

(A) implies that if

$$m_{ijk} = \inf\left\{f(x,y,z)\,\middle|\,(x,y,z)\in R_{ijk}\right\} \text{ and } M_{ijk} = \sup\left\{f(x,y)\,\middle|\,(x,y,z)\in R_{ijk}\right\},$$

then

$$\sum_{i=1}^{r} m_{ijk}(x_i - x_{i-1}) \le G(\eta_{jk}, \zeta_{jk}) \le \sum_{i=1}^{r} M_{ijk}(x_i - x_{i-1}).$$

Multiplying through by $(y_j - y_{j-1})(z_k - z_{k-1})$, summing over $1 \le j \le s$, $1 \le k \le t$, and recalling (B) yields

$$s(\mathbf{P}) \le \sigma \le S(\mathbf{P}), \tag{C}$$

where $s(\mathbf{P})$ and $S(\mathbf{P})$ are the upper and lower sums of f over $\mathbf{P}$. Now let $\widehat{s}(P_2 \times P_3)$ and $\widehat{S}(P_2 \times P_3)$ be the upper and lower sums of G over $P_2 \times P_3$; since they are respectively the infimum and supremum of the Riemann sums of G over $P_2 \times P_3$ (Theorem 7.1.5), (C) implies that

$$s(\mathbf{P}) \le \widehat{s}(P_2 \times P_3) \le \widehat{S}(P_2 \times P_3) \le S(\mathbf{P}). \tag{D}$$

Since f is integrable on R, there is for each $\epsilon > 0$ a partition $\mathbf{P}$ of R such that $S(\mathbf{P}) - s(\mathbf{P}) < \epsilon$ (Theorem 7.1.12). Consequently, from (D), there is a partition $P_2 \times P_3$ of $I_2 \times I_3$ such that $\widehat{S}(P_2 \times P_3) - \widehat{s}(P_2 \times P_3) < \epsilon$, so G is integrable on $I_2 \times I_3$ (Theorem 7.1.12).

It remains to verify that

$$\int_R f(x, y, z)\, d(x, y, z) = \int_{I_2 \times I_3} G(y, z)\, d(y, z). \tag{E}$$

From (B) and the definition of $\int_{I_2\times I_3} G(y, z)\, d(y, z)$, there is for each $\epsilon > 0$ a $\delta > 0$ such that

$$\left| \int_{I_2 \times I_3} G(y, z)\, d(y, z) - \sigma \right| < \epsilon \quad \text{if} \quad \|P_2 \times P_3\| < \delta;$$

that is,

$$\sigma - \epsilon < \int_{I_2 \times I_3} G(y, z)\, d(y, z) < \sigma + \epsilon \quad \text{if} \quad \|P_2 \times P_3\| < \delta.$$

This and (C) imply that

$$s(\mathbf{P}) - \epsilon < \int_{I_2 \times I_3} G(y, z)\, d(y, z) < S(\mathbf{P}) + \epsilon \quad \text{if} \quad \|\mathbf{P}\| < \delta,$$

and this implies that

$$\underline{\int_R} f(x, y, z)\, d(x, y, z) - \epsilon \le \int_{I_2 \times I_3} G(y, z)\, d(y, z) \le \overline{\int_R} f(x, y, z)\, d(x, y, z) + \epsilon \tag{F}$$

(Definition 7.1.4). Since

$$\underline{\int_R} f(x, y, z)\, d(x, y, z) = \overline{\int_R} f(x, y, z)\, d(x, y, z)$$

(Theorem 7.1.8) and ϵ can be made arbitrarily small, (F) implies (E).

(b) Suppose that

$$z_{k-1} \le \zeta_k \le z_k, \quad 1 \le k \le t, \tag{A}$$

so

$$\sigma = \sum_{k=1}^{t} H(\zeta_k)(z_k - z_{k-1}) \tag{B}$$

is a typical Riemann sum of H over P_3. Since

$$H(\zeta_k) = \int_{I_1 \times I_2} f(x, y, \zeta_k)\, dx = \sum_{i=1}^{r} \sum_{j=1}^{s} \int_{[x_{i-1}, x_i] \times [y_{j-1}, y_j]} f(x, y, \zeta_k)\, dx,$$

(A) implies that if

$$m_{ijk} = \inf\left\{ f(x, y, z) \,\middle|\, (x, y, z) \in R_{ijk} \right\} \text{ and } M_{ijk} = \sup\left\{ f(x, y) \,\middle|\, (x, y, z) \in R_{ijk} \right\},$$

then

$$\sum_{i=1}^{r} \sum_{j=1}^{s} m_{ijk}(x_i - x_{i-1})(y_j - y_{j-1}) \le H(\zeta_k) \le \sum_{i=1}^{r} \sum_{j=1}^{t} M_{ijk}(x_i - x_{i-1})(y_j - y_{j-1}).$$

Multiplying through by $(z_k - z_{k-1})$, summing over $1 \le k \le t$, and recalling (B) yields

$$s(\mathbf{P}) \le \sigma \le S(\mathbf{P}), \tag{C}$$

where $s(\mathbf{P})$ and $S(\mathbf{P})$ are the upper and lower sums of f over $\mathbf{P}$. Now let $\widehat{s}(P_3)$ and $\widehat{S}(P_3)$ be the upper and lower sums of H over P_3; since they are respectively the infimum and supremum of the Riemann sums of H over P_3 (Theorem 3.1.4), (C) implies that

$$s(\mathbf{P}) \le \widehat{s}(P_3) \le \widehat{S}(P_3) \le S(\mathbf{P}). \tag{D}$$

Since f is integrable on R, there is for each $\epsilon > 0$ a partition $\mathbf{P}$ of R such that $S(\mathbf{P}) - s(\mathbf{P}) < \epsilon$ (Theorem 7.1.12). Consequently, from (D), there is a partition of I_3 such that $\widehat{S}(P_3) - \widehat{s}(P_3) < \epsilon$, so H is integrable on I_3 (Theorem 3.2.7).

It remains to verify that

$$\int_R f(x, y, z)\, d(x, y, z) = \int_{a_3}^{b_3} H(z)\, dz. \tag{E}$$

From (B) and the definition of $\int_{a_3}^{b_3} H(z)\, dz$, there is for each $\epsilon > 0$ a $\delta > 0$ such that

$$\left| \int_{a_3}^{b_3} H(z)\, dz - \sigma \right| < \epsilon \quad \text{if} \quad \|P_3\| < \delta;$$

that is,

$$\sigma - \epsilon < \int_{a_3}^{b_3} H(z)\, dz < \sigma + \epsilon \quad \text{if} \quad \|P_3\| < \delta.$$

This and (C) imply that

$$s(\mathbf{P})-\epsilon<\int_{a_3}^{b_3} H(z)\,dz<S(\mathbf{P})+\epsilon \quad \text{if} \quad \|\mathbf{P}\|<\delta,$$

and this implies that

$$\underline{\int_R} f(x,y,z)\,d(x,y,z)-\epsilon \le \int_{a_3}^{b_3} H(z)\,dz \le \overline{\int_R} f(x,y,z)\,d(x,y,z)+\epsilon \qquad \text{(F)}$$

(Definition 7.1.4). Since

$$\underline{\int_R} f(x,y,z)\,d(x,y,z)=\overline{\int_R} f(x,y,z)\,d(x,y,z)$$

(Theorem 7.1.8) and ϵ can be made arbitrarily small, (F) implies (E).

7.2.3. If $\epsilon>0$, there is a $\delta>0$ such that $|f(x,y)-f(x,y')|<\epsilon$ if $|y-y'|<\delta$ (Theorem 5.2.14). Therefore, $|F(y)-F(y')|\le\int_a^b|f(x,y)-f(x,y')|\,dx<\epsilon(b-a)$ if $|y-y'|<\delta$.

7.2.4. Let $P_1: a=x_0<x_1<\cdots<x_r=b$ and $P_2: c=y_0<y_1<\cdots<y_s=d$ be partitions of $[a,b]$ and $[c,d]$, and $\mathbf{P}=P_1\times P_2$. Then

$$\begin{aligned}
S(\mathbf{P})-s(\mathbf{P}) &= \sum_{j=1}^{s}\sum_{i=1}^{r}(f(x_i,y_j)-f(x_{i-1},y_{j-1}))(x_i-x_{i-1})(y_j-y_{j-1})\\
&= \sum_{j=1}^{s}\left[\sum_{i=1}^{r}(f(x_i,y_j)-f(x_{i-1},y_j))(x_i-x_{i-1})\right](y_j-y_{j-1})\\
&\quad+\sum_{i=1}^{r}\left[\sum_{j=1}^{s}(f(x_{i-1},y_j)-f(x_{i-1},y_{j-1}))(y_j-y_{j-1})\right](x_i-x_{i-1})\\
&\le \|P_1\|\sum_{j=1}^{s}(f(b,y_j)-f(a,y_j))(y_j-y_{j-1})\\
&\quad+\|P_2\|\sum_{i=1}^{r}(f(x_{i-1},d)-f(x_{i-1},c))(x_i-x_{i-1})\\
&\le \|\mathbf{P}\|(f(b,d)-f(a,c))(d-c+b-a).
\end{aligned}$$

Since we can make $\|\mathbf{P}\|$ arbitrarily small, this and Theorem 7.1.12 imply that $\int_R f(x,y)\,d(x,y)$ exists. For each fixed y, $f(x,y)$ is monotonic in x, so Theorem 3.2.9 implies that $\int_a^b f(x,y)\,dx$ exists, $c\le y\le d$.

7.2.5. **(a)**

$$\begin{aligned}\int_R (xy+1)\,d(x,y) &= \int_1^2 dy \int_0^1 (xy+1)\,dx \\ &= \int_1^2 \left[\left(\frac{x^2y}{2}+x\right)\Big|_{x=0}^{1}\right] dy \\ &= \int_1^2 \left(\frac{y}{2}+1\right) dy = \left(\frac{y^2}{4}+y\right)\Big|_1^2 = \frac{7}{4}.\end{aligned}$$

(b)

$$\begin{aligned}\int_R (2x+3y)\,d(x,y) &= \int_1^2 dy \int_1^3 (2x+3y)\,dx \\ &= \int_1^2 \left[(x^2+3xy)\Big|_{x=1}^{3}\right] dy \\ &= \int_1^2 (6y+8)\,dy = (3y^2+8y)\Big|_1^2 = 17\end{aligned}$$

(c)

$$\begin{aligned}\int_R \frac{xy}{\sqrt{x^2+y^2}}\,d(x,y) &= \int_0^1 y\,dy \int_0^1 \frac{x}{\sqrt{x^2+y^2}}\,dx \\ &= \int_0^1 y\left[\sqrt{x^2+y^2}\Big|_{x=0}^{1}\right] dy \\ &= \int_0^1 \left(y\sqrt{y^2+1}-y^2\right) dy \\ &= \frac{1}{3}\left((y^2+1)^{3/2}-y^3\right)\Big|_0^1 = \frac{2(\sqrt{2}-1)}{3}.\end{aligned}$$

(d)

$$\begin{aligned}\int_R x\cos xy\cos 2\pi x\,d(x,y) &= \int_0^{\pi/4} \cos 2\pi x\,dx \int_0^{2\pi} x\cos xy\,dy \\ &= \int_0^{\pi/4} \cos 2\pi x\left[\sin xy\Big|_{y=0}^{2\pi}\right] dx \\ &= \int_0^{\pi/4} \cos 2\pi x \sin 2\pi x\,dx \\ &= \frac{\sin^2 2\pi x}{4\pi}\Big|_0^{\pi/4} = \frac{1}{4\pi}.\end{aligned}$$

7.2.6. If $\epsilon > 0$ choose m so that $2^{-m+1/2} < \epsilon$. Then, for arbitrary (x,y), choose odd integers p and q so that $|2^m x - p| \le 1$ and $|2^m y - q| \le 1$. Then

$$\left[(x-2^{-m}p)^2 + (y-2^{-m}q)^2\right]^{1/2} < \epsilon,$$

so A is dense in $\mathbb{R}^2$. If $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ is a partition of $R = [a,b]\times[c,d]$, then $R_j \cap A$ and $R_j \cap A^c$ are both nonempty, $1 \le j \le k$. Hence, $s(\mathbf{P}) = 0$, while $S(\mathbf{P}) = (b-a)(d-c)$. Therefore, $\underline{\int_R} f(x,y)\,d(x,y) = 0$, while $\overline{\int_R} f(x,y)\,d(x,y) = (b-a)(d-c)$, so f is not integrable on R (Theorem 7.1.8).

Since the set of point $\{2^{-k}r \mid k = \text{nonnegative integer},\ r = \text{odd integer}\}$ is denumerable, $f(x,y) = 1$ on any horizontal line except on a set of values of x with measure zero in $\mathbb{R}$ (Example 3.5.3); therefore, for a fixed y, $f(x,y)$ is integrable on $[a,b]$ and $\int_a^b f(x,y)\,dx = b-a$, by Theorem 3.5.6 and Exercise 3.5.6. Therefore, $\int_c^d dy \int_a^b f(x,y)\,dx = (d-c)(b-a)$. Similarly, $f(x,y) = 1$ on any vertical line except on a set of values of y with measure zero in $\mathbb{R}$; therefore, for a fixed x, $f(x,y)$ is integrable on $[c,d]$ and $\int_c^d f(x,y)\,dy = d-c$. Therefore, $\int_a^b dx \int_c^d f(x,y)\,dy = (b-a)(d-c)$.

7.2.7. Recall that the rationals and irrationals are both dense on the real line.

(a) Let $P_1 : 0 = x_0 < x_1 < \cdots < x_s = 1$ and $P_2 : 0 = y_0 < y_1 < \cdots < y_t = 1$ be partitions of $[0,1]$, and let $\mathbf{P} = P_1 \times P_2$. Let $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. From Lemma 7.1.6, we may assume that $x_r = 1/2$ for some integer r, with $0 < r < s$. Then

$$\inf\{f(x,y) \mid (x,y) \in R_{ij}\} = \begin{cases} 2x_{i-1}y_{j-1}, & 1 \le i \le r, \\ y_{j-1}, & r+1 \le i \le s, \end{cases}$$

$$\sup\{f(x,y) \mid (x,y) \in R_{ij}\} = \begin{cases} y_j, & 1 \le i \le r, \\ 2x_i y_j, & r+1 \le i \le s. \end{cases}$$

$$\begin{aligned} s(\mathbf{P}) &= \left(2\sum_{i=1}^{r} x_{i-1}(x_i - x_{i-1}) + \sum_{i=r+1}^{s}(x_i - x_{i-1})\right)\sum_{j=1}^{t} y_{j-1}(y_j - y_{j-1}) \\ &= \to \left(2\int_0^{1/2} x\,dx + \int_{1/2}^1 dx\right)\int_0^1 y\,dy = \frac{3}{8} \quad \text{as} \quad \|\mathbf{P}\| \to 0, \end{aligned}$$

$$\begin{aligned} S(\mathbf{P}) &= \left(\sum_{i=1}^{r}(x_i - x_{i-1}) + 2\sum_{i=r+1}^{s} x_i(x_i - x_{i-1})\right)\sum_{j=1}^{t} y_j(y_j - y_{j-1}) \\ &= \to \left(\int_0^{1/2} dx + 2\int_{1/2}^1 x\,dx\right)\int_0^1 y\,dy = \frac{5}{8} \quad \text{as} \quad \|\mathbf{P}\| \to 0, \end{aligned}$$

so $\underline{\int_R} f(x,y)\,d(x,y) = \frac{3}{8}$ and $\overline{\int_R} f(x,y)\,d(x,y) = \frac{5}{8}$.

(b) Let $P : 0 = y_0 < y_1 < \cdots < y_t = 1$ be a partition of $[0,1]$. For a given x, let $s_x(P)$ and $S_x(P)$ be lower and upper sums of $f(x,y)$ over P. Since

$$\inf\{f(x,y) \mid y_{j-1} \le y \le y_j\} = \begin{cases} 2xy_{j-1}, & 0 \le x \le 1/2, \\ y_{j-1}, & 1/2 \le x \le 1, \end{cases}$$

$$\sup\{f(x,y) \mid y_{j-1} \le y \le y_j\} = \begin{cases} y_j, & 0 \le x \le 1/2, \\ 2xy_j, & 1/2 \le x \le 1, \end{cases}$$

$$s_x(P) = \begin{cases} 2x\displaystyle\sum_{j=1}^{t} y_{j-1}(y_j - y_{j-1}) \to 2x\int_0^1 y\,dy = x \text{ as } \|P\| \to 0, & 0 \le x \le 1/2 \\ \displaystyle\sum_{j=1}^{t} y_{j-1}(y_j - y_{j-1}) \to \int_0^1 y\,dy = 1/2 \text{ as } \|P\| \to 0, & 1/2 \le x \le 1, \end{cases}$$

$$S_x(P) = \begin{cases} \displaystyle\sum_{j=1}^{t} y_j(y_j - y_{j-1}) \to \int_0^1 y\,dy = 1/2 \text{ as } \|P\| \to 0, & 0 \le x \le 1/2 \\ 2x\displaystyle\sum_{j=1}^{t} y_j(y_j - y_{j-1}) \to 2x\int_0^1 y\,dy = x \text{ as } \|P\| \to 0, & 1/2 \le x \le 1; \end{cases}$$

$$\underline{\int_0^1} f(x,y)\,dy = \begin{cases} x, & 0 \le x \le 1/2, \\ 1/2, & 1/2 \le x \le 1, \end{cases} \text{ and } \overline{\int_0^1} f(x,y)\,dy = \begin{cases} 1/2, & 0 \le x \le 1/2, \\ x, & 1/2 \le x \le 1, \end{cases}$$

so $\displaystyle\int_0^1 \left(\underline{\int_0^1} f(x,y)\,dy\right) dx = \frac{3}{8}$ and $\displaystyle\int_0^1 \left(\overline{\int_0^1} f(x,y)\,dy\right) dx = \frac{5}{8}$.

7.2.8. Recall that the rationals and irrationals are both dense on the real line.

Let $P_1 : 0 = x_0 < x_1 < \cdots < x_s = 1$, $P_2 : 0 = y_0 < y_1 < \cdots < y_t = 1$, and $P_3 : 0 = z_0 < z_1 < \cdots < z_u = 1$ be partitions of $[0,1]$, We exploit the identities $\displaystyle\sum_{j=1}^{t}(y_j - y_{j-1}) = \sum_{k=1}^{u}(z_k - z_{k-1}) = 1$ and $\displaystyle\sum_{i=1}^{r}(x_i - x_{i-1}) = \sum_{i=r+1}^{s}(x_i - x_{i-1}) = \frac{1}{2}$ without specific references.

(a) Let $\mathbf{P} = P_1 \times P_2 \times P_3$ and $R_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. From Lemma 7.1.6, we may assume that $x_r = 1/2$ for some integer r, with $0 < r < s$. Then

$$\inf\left\{f(x,y,z) \,\middle|\, (x,y,z) \in R_{ijk}\right\} = \begin{cases} 2x_{i-1}(y_{j-1} + z_{k-1}), & 1 \le i \le r, \\ y_{j-1} + z_{k-1}, & r+1 \le i \le s, \end{cases}$$

$$\sup\left\{f(x,y,z) \,\middle|\, (x,y,z) \in R_{ijk}\right\} = \begin{cases} y_j + z_k, & 1 \le i \le r, \\ 2x_i(y_j + z_k), & r+1 \le i \le s, \end{cases}$$

In the following we exploit the identities $\displaystyle\sum_{j=1}^{t}(y_j - y_{j-1}) = \sum_{k=1}^{u}(z_k - z_{k-1}) = 1$ and

$\sum_{i=1}^{r}(x_i - x_{i-1}) = \sum_{i=r+1}^{s}(x_i - x_{i-1}) = \frac{1}{2}$ without specific references.

$$\begin{aligned}
s(P) &= 2\sum_{i=1}^{r}\sum_{j=1}^{t}\sum_{k=1}^{u} x_{i-1}(y_{j-1}+z_{k-1})(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\\
&\quad + \sum_{i=r+1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{u}(y_{j-1}+z_{k-1})(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\\
&= 2\left(\sum_{j=1}^{t} y_{j-1}(y_j - y_{j-1}) + \sum_{k=1}^{u} z_{k-1}(z_k - z_{k-1})\right)\sum_{i=1}^{r} x_{i-1}(x_i - x_{i-1})\\
&\quad +\frac{1}{2}\left(\sum_{j=1}^{t} y_{j-1}(y_j - y_{j-1}) + \sum_{k=1}^{u} z_{k-1}(z_k - z_{k-1})\right)\\
&\to \left(\int_0^1 y\,dy + \int_0^1 z\,dz\right)\left(2\int_0^{1/2} x\,dx + \frac{1}{2}\right) = \frac{3}{4} \text{ as } \|\mathbf{P}\| \to 0.
\end{aligned}$$

$$\begin{aligned}
S(P) &= \sum_{i=1}^{r}\sum_{j=1}^{t}\sum_{k=1}^{u}(y_j + z_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\\
&\quad +2\sum_{i=r+1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{u} x_i(y_j + z_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\\
&= \frac{1}{2}\left(\sum_{j=1}^{t} y_j(y_j - y_{j-1}) + \sum_{k=1}^{u} z_k(z_k - z_{k-1})\right)\\
&\quad +2\left(\sum_{j=1}^{t} y_j(y_j - y_{j-1}) + \sum_{k=1}^{u} z_k(z_k - z_{k-1})\right)\sum_{i=1}^{r} x_i(x_i - x_{i-1})\\
&\to \left(\int_0^1 y\,dy + \int_0^1 z\,dz\right)\left(\frac{1}{2} + 2\int_{1/2}^{1} x\,dx+\right) = \frac{5}{4} \text{ as } \|\mathbf{P}\| \to 0.
\end{aligned}$$

Therefore, $\underline{\int_R} f(x,y,z)\,d(x,y,z) = \frac{3}{4}$ and $\overline{\int_R} f(x,y,z)\,d(x,y,z) = \frac{5}{4}$.

(b) Let $\widetilde{\mathbf{P}} = P_1 \times P_2$ and $\widetilde{R}_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. From Lemma 7.1.6, we may assume that $x_r = 1/2$ for some integer r, with $0 < r < s$. For a given z, let $s_z(\widetilde{P})$ and

$S_z(\widetilde{P})$ be lower and upper sums of $f(x,y,z)$ over $\widetilde{P}$. Since

$$\inf\left\{f(x,y,z)\,\middle|\,(x,y)\in\widetilde{R}_{ij}\right\}=\begin{cases}2x_{i-1}(y_{j-1}+z), & 1\le i\le r,\\ y_{j-1}+z, & r+1\le i\le s,\end{cases}$$

$$\sup\left\{f(x,y,z)\,\middle|\,(x,y)\in\widetilde{R}_{ij}\right\}=\begin{cases}y_j+z, & 1\le i\le r,\\ 2x_i(y_j+z), & r+1\le i\le s,\end{cases}$$

$$\begin{aligned}
s_z(\widetilde{P}) &= 2\sum_{i=1}^{r}\sum_{j=1}^{t}x_{i-1}(y_{j-1}+z)(x_i-x_{i-1})(y_j-y_{j-1})\\
&\quad+\sum_{i=r+1}^{s}\sum_{j=1}^{t}(y_{j-1}+z)(x_i-x_{i-1})(y_j-y_{j-1})\\
&= 2\left(\sum_{i=1}^{r}x_{i-1}(x_i-x_{i-1})\right)\left(\sum_{j=1}^{t}y_{j-1}(y_j-y_{j-1})\right)+2z\sum_{i=1}^{r}x_{i-1}(x_i-x_{i-1})\\
&\quad+\frac{1}{2}\left(z+\sum_{j=1}^{t}y_{j-1}(y_j-y_{j-1})\right)\\
&= \left(\frac{1}{2}+2\sum_{i=1}^{r}x_{i-1}(x_i-x_{i-1})\right)\left(z+\sum_{j=1}^{t}y_{j-1}(y_j-y_{j-1})\right)\\
&\quad\to\left(\frac{1}{2}+2\int_0^{1/2}x\,dx\right)\left(z+\int_0^1 y\,dy\right)=\frac{3}{4}\left(z+\frac{1}{2}\right)\ \text{as}\ \|\mathbf{P}\to 0\|
\end{aligned}$$

$$\begin{aligned}
S_z(\widetilde{P}) &= \sum_{i=1}^{r}\sum_{j=1}^{t}(y_j+z)(x_i-x_{i-1})(y_j-y_{j-1})\\
&\quad+2\sum_{i=r+1}^{s}\sum_{j=1}^{t}x_i(y_j+z)(x_i-x_{i-1})(y_j-y_{j-1})\\
&= \frac{1}{2}\left(z+\sum_{j=1}^{t}y_j(y_j-y_{j-1})\right)\\
&\quad+2\left(\sum_{i=r+1}^{s}x_i(x_i-x_{i-1})\right)\left(\sum_{j=1}^{t}y_j(y_j-y_{j-1})\right)+2z\sum_{i=r+1}^{s}x_i(x_i-x_{i-1})\\
&= \left(\frac{1}{2}+2\sum_{i=r+1}^{s}x_i(x_i-x_{i-1})\right)\left(z+\sum_{j=1}^{t}yj(y_j-y_{j-1})\right)\\
&\quad\to\left(\frac{1}{2}+2\int_{1/2}^{1}x\,dx\right)\left(z+\int_0^1 y\,dy\right)=\frac{5}{4}\left(z+\frac{1}{2}\right)\ \text{as}\ \|\mathbf{P}\to 0\|,
\end{aligned}$$

so $\underline{\int}_{\widetilde{R}} f(x,y,z)\,d(x,y) = \frac{3}{4}\left(z+\frac{1}{2}\right)$ and $\overline{\int}_{\widetilde{R}} f(x,y,z)\,d(x,y) = \frac{5}{4}\left(z+\frac{1}{2}\right)$.

(c) $\int_0^1 f(x,y,z)\,dx = y+z$ for all x;

$$\int_0^y dy\int_0^1 f(x,y,z)\,dx = \int_0^1 (y+z)\,dy = \left(\frac{y^2}{2}+yz\right)\Big|_0^1 = \frac{1}{2}+z;$$

$$\int_0^1 dz\int_0^y dy\int_0^1 f(x,y,z)\,dx = \int_0^1\left(\frac{1}{2}+z\right)dz = \left(\frac{z}{2}+\frac{z^2}{2}\right)\Big|_0^1 = 1.$$

7.2.9. Let $a = x_0 < x_1 < \cdots < x_r = b$ and $c = y_0 < y_1 < \cdots < y_s = b$ be partitions of $[a,b]$ and $[c,d]$.

(a) From Exercise 3.2.6(**a**) with $g(x) = \underline{\int_c^d} f(x,y)\,dy$,

$$\underline{\int_a^b}\left(\underline{\int_c^d} f(x,y)\,dy\right)dx = \sum_{i=1}^r \underline{\int_{x_{i-1}}^{x_i}}\left(\underline{\int_c^d} f(x,y)\,dy\right)dx \tag{A}$$

From Exercise 3.2.6(**a**) with $g(y) = f(x,y)$ (x fixed), $\underline{\int_c^d} f(x,y)\,dy = \sum_{j=1}^s F_j(x)$, with $F_j(x) = \underline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy$. Since

$$\inf\left\{\sum_{j=1}^s F_j(x)\,\Big|\,x_{j-1}\le x\le x_j\right\} \ge \sum_{j=1}^s \inf\left\{F_j(x)\,\big|\,x_{j-1}\le x\le x_j\right\},$$

(A) implies that

$$\underline{\int_a^b}\left(\underline{\int_c^d} f(x,y)\,dy\right)dx \ge \sum_{i=1}^r\sum_{j=1}^s \underline{\int_{x_{i-1}}^{x_i}}\left(\underline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy\right)dx. \tag{B}$$

Since

$$\underline{\int_{x_{i-1}}^{x_i}}\left(\underline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy\right)dx \ge m_{ij}(x_i-x_{i-1})(y_j-y_{j-1})$$

with

$$m_{ij} = \inf\left\{f(x,y)\,\big|\,x_{i-1}\le x\le x_i,\ y_{j-1}\le y\le y_j\right\},$$

(B) implies that $\underline{\int_a^b}\left(\underline{\int_c^d} f(x,y)\,dy\right)dx$ is an upper bound for all lower sums of f over partitions of $R = [a,b]\times[c,d]$. Since $\underline{\int_R} f(x,y)\,d(x,y)$ is the supremum of this set, $\underline{\int_R} f(x,y)\,d(x,y) \le \underline{\int_a^b}\left(\underline{\int_c^d} f(x,y)\,dy\right)dx$.

(b) From Exercise 3.2.6**(b)** with $g(x) = \overline{\int_c^d} f(x,y)\,dy$,

$$\overline{\int_a^b} \left(\overline{\int_c^d} f(x,y)\,dy \right) dx = \sum_{i=1}^{r} \overline{\int_{x_{i-1}}^{x_i}} \left(\overline{\int_c^d} f(x,y)\,dy \right) dx \qquad \text{(A)}$$

From Exercise 3.2.6**(b)** with $g(y) = f(x,y)$ (x fixed), $\overline{\int_c^d} f(x,y)\,dy = \sum_{j=1}^{s} F_j(x)$, with $F_j(x) = \overline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy$. Since

$$\sup\left\{ \sum_{j=1}^{s} F_j(x) \,\Big|\, x_{j-1} \le x \le x_j \right\} \le \sum_{j=1}^{s} \sup\left\{ F_j(x) \,\big|\, x_{j-1} \le x \le x_j \right\},$$

(A) implies that

$$\overline{\int_a^b} \left(\overline{\int_c^d} f(x,y)\,dy \right) dx \le \sum_{i=1}^{r} \sum_{j=1}^{s} \overline{\int_{x_{i-1}}^{x_i}} \left(\overline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy \right) dx. \qquad \text{(B)}$$

Since

$$\overline{\int_{x_{i-1}}^{x_i}} \left(\overline{\int_{y_{j-1}}^{y_j}} f(x,y)\,dy \right) dx \le M_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

with

$$M_{ij} = \sup\left\{ f(x,y) \,\big|\, x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j \right\},$$

(B) implies that $\overline{\int_a^b} \left(\overline{\int_c^d} f(x,y)\,dy \right) dx$ is a lower bound for all upper sums of f over partitions of $R = [a,b] \times [c,d]$. Since $\overline{\int_R} f(x,y)\,d(x,y)$ is the infimum of this set, $\overline{\int_R} f(x,y)\,d(x,y) \ge \overline{\int_a^b} \left(\overline{\int_c^d} f(x,y)\,dy \right) dx$.

7.2.10. Let $U(x) = \overline{\int_a^b} f(x,y)\,dy$ and $L(x) = \underline{\int_c^d} f(x,y)\,dy$. From Exercise 7.2.9, (A) $\overline{\int_a^b} U(x)\,dx \le \overline{\int_R} f(x,y)\,d(x,y)$ and (B) $\underline{\int_a^b} L(x)\,dx \ge \underline{\int_R} f(x,y)\,d(x,y)$. Since $L(x) \le U(x)$ (Theorem 7.1.7), (A) implies that (C) $\overline{\int_a^b} L(x)\,dx \le \overline{\int_R} f(x,y)\,d(x,y)$ and (B) implies that (D) $\underline{\int_a^b} U(x)\,dx \ge \underline{\int_R} f(x,y)\,d(x,y)$. Since $\overline{\int_R} f(x,y)\,d(x,y) =$

$\underline{\int_R} f(x,y)\,d(x,y) = \int_R f(x,y)\,d(x,y)$ (Theorem 7.1.8 and $\underline{\int_a^b} U(x)\,dx \le \overline{\int_a^b} U(x)\,dx$, (A) and (D) imply that (E) $\underline{\int_a^b} U(x)\,dx = \overline{\int_a^b} U(x)\,dx = \int_R f(x,y)\,d(x,y)$, and, since $\underline{\int_a^b} L(x)\,dx \le \overline{\int_a^b} L(x)\,dx$, (B) and (C) imply that (F) $\underline{\int_a^b} L(x)\,dx = \overline{\int_a^b} L(x)\,dx = \int_R f(x,y)\,d(x,y)$. From Theorem 3.2.6, (E) and (F) imply that U and L are integrable on $[a,b]$, and $\int_a^b L(x)\,dx = \int_a^b U(x)\,dx = \int_R f(x,y)\,d(x,y)$,

7.2.11. **(a)**

$$\begin{aligned}
\int_R (x-2y+3z)\,d(x,y,z) &= \int_{-3}^{2} dz \int_2^5 dy \int_{-2}^{0} (x-2y+3z)\,dx \\
&= \int_{-3}^{2} dz \int_2^5 \left[\left(\frac{x^2}{2} - 2xy + 3xz \right) \Bigg|_{x=-2}^{0} \right] dy \\
&= \int_{-3}^{2} dz \int_2^5 (-4y + 6z - 2)\,dy \\
&= \int_{-3}^{2} \left[(-2y^2 + 6yz - 2y) \Bigg|_{y=-2}^{5} \right] dz \\
&= \int_{-3}^{2} (8z - 48)\,dz = (9z^2 - 48z)\Bigg|_{-3}^{2} = -285.
\end{aligned}$$

(b) $\displaystyle\int_R e^{-x^2-y^2} \sin x \sin z \, d(x,y,z) = \int_0^{\pi/2} \sin z\,dz \int_0^2 e^{-y^2}\,dy \int_{-1}^{1} e^{-x^2} \sin x\,dx = 0.$

(c)
$$\begin{aligned}
\int_R (xy + 2xz + yz)\,d(x,y,z) &= \int_0^1 dy \int_{-1}^{1} dz \int_{-1}^{1} (xy + 2xz + yz)\,dx \\
&= \int_0^1 2y\,dy \int_{-1}^{1} z\,dz = 0.
\end{aligned}$$

(d)

$$\begin{aligned}
\int_R x^2y^3ze^{xy^2z^2}\,d(x,y,z) &= \int_0^1 dx\int_0^1 xy\,dy\int_0^1 xy^2ze^{xy^2z^2}\,dz \\
&= \frac{1}{2}\int_0^1 dx\int_0^1 xy\left[e^{xy^2z^2}\Big|_{z=0}^{1}\right]dy \\
&= \frac{1}{2}\int_0^1 dx\int_0^1 xy\left(e^{xy^2}-1\right)dy \\
&= \frac{1}{4}\int_0^1\left[\left(e^{xy^2}-xy^2\right)\Big|_{y=0}^{1}\right]dy \\
&= \int_0^1 (e^x-x-1)\,dy = \frac{1}{4}\left(e^x-\frac{x^2}{2}-x\right)\Big|_0^1 \\
&= \frac{1}{4}\left(e-\frac{5}{2}\right).
\end{aligned}$$

7.2.12. **(a)**

$$\begin{aligned}
\int_S (2x+y^2)\,d(x,y) &= \int_{-3}^{3} dy\int_0^{9-y^2}(2x+y^2)\,dx \\
&= \int_{-3}^{3}\left[(x^2+xy^2)\Big|_{x=0}^{9-y^2}\right]dy \\
&= 9\int_{-3}^{3}(9-y^2)\,dy = 9\left(9y-\frac{y^3}{3}\right)\Big|_{-3}^{3} = 324.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_S 2xy\,d(x,y) &= 2\int_0^1 y\,dy\int_{x=y^2}^{\sqrt{y}} x\,dx = \int_0^1 y\left[x^2\Big|_{x=y^2}^{x=\sqrt{y}}\right]dy \\
&= \int_0^1 y(y-y^4)\,dy = \left(\frac{y^3}{3}-\frac{y^6}{6}\right)\Big|_0^1 = \frac{1}{6}.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_S e^x\frac{\sin y}{y}\,d(x,y) &= \int_{\pi/2}^{\pi}\frac{\sin y}{y}\,dy\int_{\log y}^{\log 2y} e^x\,dx \\
&= \int_{\pi/2}^{\pi}\frac{\sin y}{y}\left[e^x\Big|_{x=\log y}^{\log 2y}\right]dy \\
&= \int_{\pi/2}^{\pi}\sin y\,dy = -\cos y\Big|_{\pi/2}^{\pi} = 1.
\end{aligned}$$

7.2.13. The curves $y = x^2$ and $y = 2x$ intersect at $(0, 0)$ and $(2, 4)$.

$$\int_S (x+y)\,d(x,y) = \int_0^4 dy \int_{y/2}^{\sqrt{y}} (x+y)\,dx = \int_0^4 \left[\left(\frac{x^2}{2} + xy\right)\Bigg|_{x=y/2}^{\sqrt{y}}\right] dy$$

$$= \int_0^4 \left(y^{3/2} + \frac{y}{2} - \frac{5y^2}{8}\right) dy = \left(-\frac{5y^3}{24} + \frac{2y^{5/2}}{5} + \frac{y^2}{4}\right)\Bigg|_0^4 = \frac{52}{15}.$$

$$\int_S (x+y)\,d(x,y) = \int_0^2 dx \int_{x^2}^{2x} (x+y)\,dy = \int_0^2 \left[\left(xy + \frac{y^2}{2}\right)\Bigg|_{y=x^2}^{2x}\right] dx$$

$$= \int_0^2 \left(4x^2 - x^3 - \frac{x^4}{2}\right) dx = \left(\frac{4x^3}{3} - \frac{x^4}{4} - \frac{x^5}{10}\right)\Bigg|_0^2 = \frac{52}{15}.$$

7.2.14. **(a)** $A = \int_{-1}^1 dx \int_{x^2-9}^{x^2+9} dy = 18 \int_{-1}^1 dx = 36.$

(b) $A = \int_0^1 dx \int_{x+2}^{4-x} dy = 2\int_0^1 (1-x)\,dx = -(1-x)^2\Big|_0^1 = 1$

(c) $A = 2\int_0^2 dy \int_{y^2-4}^{4-y^2} dx = 4\int_0^2 (4-y^2)\,dy = 4\left(4y - \frac{y^3}{3}\right)\Bigg|_0^2 = \frac{64}{3}.$

(d) $A = \int_0^3 dx \int_{-2x}^{e^{2x}} dy = \int_0^3 (e^{2x} + 2x)\,dx = \left(\frac{e^{2x}}{2} + x^2\right)\Bigg|_0^3 = \frac{e^6 + 17}{2}.$

7.2.16.
$$\begin{aligned}
\int_S f(x,y,z)\,d(x,y,z) &= \int_0^{1/3} dz \int_0^{(1-3z)/2} dy \int_0^{1-2y-3z} f(x,y,z)\,dx \\
&= \int_0^{1/2} dy \int_0^{(1-2y)/3} dz \int_0^{1-2y-3z} f(x,y,z)\,dx \\
&= \int_0^{1/3} dz \int_0^{1-3z} dx \int_0^{(1-x-3z)/2} f(x,y,z)\,dy \\
&= \int_0^{1} dx \int_0^{(1-x)/3} dz \int_0^{(1-x-3z)/2} f(x,y,z)\,dy \\
&= \int_0^{1} dx \int_0^{(1-x)/2} dy \int_0^{(1-x-2y)/3} f(x,y,z)\,dz \\
&= \int_0^{1/2} dy \int_0^{1-2y} dx \int_0^{(1-x-2y)/3} f(x,y,z)\,dz
\end{aligned}$$

7.2.17. **(a)**
$$\int_S x\,d(x,y,z) = \int_0^{2/3} x\,dx \int_0^{2-3x} (2-y-3x)\,dy \int_0^{2-y-3x} dz$$
$$= \int_0^{2/3} x\,dx \int_0^{2-3x} dy$$
$$= -\frac{1}{2}\int_0^{2/3} \left[(2-y-3x)^2 \Big|_{y=0}^{2-3x} \right] dx$$
$$= \frac{1}{2}\int_0^{2/3} x(2-3x)^2\,dx$$
$$= \frac{1}{2}\int_0^{2/3} (9x^3-12x^2+4x)\,dx$$
$$= \left(\frac{9x^4}{8} - 2x^3 + x^2\right)\Big|_0^{2/3} = \frac{2}{27}.$$

(b)
$$\int_S ye^z\,d(x,y,z) = \int_0^1 dx \int_0^{\sqrt{x}} y\,dy \int_0^{y^2} e^z\,dz$$
$$= \int_0^1 dx \int_0^{\sqrt{x}} y \left[e^z \Big|_{z=0}^{y^2} \right] dy$$
$$= \int_0^1 dx \int_0^{\sqrt{x}} y(e^{y^2}-1)\,dy$$
$$= \frac{1}{2}\int_0^1 \left(e^{y^2} - y^2\right)\Big|_{y=0}^{\sqrt{x}} dx$$
$$= \frac{1}{2}\int_0^1 (e^x - x - 1)\,dx$$
$$= \frac{1}{2}\left(e^x - \frac{x^2}{2} - x\right)\Big|_0^1 = \frac{1}{2}\left(e - \frac{5}{2}\right).$$

(c)
$$\int_S xyz\,d(x,y,z) = \int_0^1 y\,dy \int_0^{\sqrt{1-y^2}} x\,dx \int_0^{\sqrt{x^2+y^2}} z\,dz$$
$$= \frac{1}{2}\int_0^1 y\,dy \int_0^{\sqrt{1-y^2}} x \left[z^2 \Big|_{z=0}^{\sqrt{x^2+y^2}} \right] dx$$
$$= \frac{1}{2}\int_0^1 y\,dy \int_0^{\sqrt{1-y^2}} (x^3 + xy^2)\,dx$$
$$= \frac{1}{4}\int_0^1 y \left[\left(\frac{x^4}{2} + x^2y^2\right)\Big|_{x=0}^{\sqrt{1-y^2}} \right] dy$$
$$= \frac{1}{8}\int_0^1 (y - y^5)\,dy = \frac{1}{8}\left(\frac{y^2}{2} - \frac{y^6}{6}\right)\Big|_0^1 = \frac{1}{24}.$$

(d)

$$\begin{aligned}\int_S yz\,d(x,y,z) &= \int_0^1 z\,dz\int_0^z y\,dy\int_{z^2}^{\sqrt{z}} dx\\ &= \int_0^1 z(\sqrt{z}-z^2)\,dz\int_0^z y\,dy\\ &= \frac{1}{2}\int_0^1 (z^{7/2}-z^5)\,dz\\ &= \frac{1}{2}\left(\frac{2z^{9/2}}{9}-\frac{z^6}{6}\right)\Big|_0^1 = \frac{1}{36}.\end{aligned}$$

7.2.18. **(a)** The two surfaces intersect on the circle $\{(x,y,z)\,|\,x^2+y^2=4,\ z=2\}$;

$$\begin{aligned}V &= \int_{-2}^2 dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\int_{x^2+y^2}^{8-x^2-y^2} dz = 2\int_{-2}^2 dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-x^2-y^2)\,dy\\ &= 4\int_{-2}^2 dx\int_0^{\sqrt{4-x^2}}(4-x^2-y^2)\,dy = 4\int_{-2}^2\left[\left(4y-x^2y-\frac{y^3}{3}\right)\Big|_{y=0}^{\sqrt{4-x^2}}\right]dx\\ &= \frac{16}{3}\int_0^2 (4-x^2)^{3/2}\,dx.\end{aligned}$$

(b)

$$\begin{aligned}V &= \int_0^1 dx\int_0^{1-x} dy\int_0^{x^2+y^2} dz = \int_0^1 dx\int_0^{1-x}(x^2+y^2)\,dy\\ &= \int_0^1\left[\left(x^2y+\frac{y^3}{3}\right)\Big|_{y=0}^{1-x}\right]dx = \int_0^1\left(\frac{1}{3}-x+2x^2-\frac{4x^3}{3}\right)dx\\ &= \left(\frac{x}{3}-\frac{x^2}{2}+\frac{2x^3}{3}-\frac{x^4}{3}\right)\Big|_0^1 = \frac{1}{6}.\end{aligned}$$

(c)

$$\begin{aligned}V &= \int_0^2 dx\int_0^{x^2} dy\int_0^{y^2} dz = \int_0^2 dx\int_0^{x^2} y^2\,dy\\ &= \frac{1}{3}\int_0^2\left[y^3\Big|_{y=0}^{x^2}\right]dx = \frac{1}{3}\int_0^2 x^6\,dx = \frac{1}{21}x^7\Big|_0^2 = \frac{128}{21}.\end{aligned}$$

(d)

$$\begin{aligned}V &= \int_0^1 dx\int_0^{\sqrt{1-x^2}} d\theta\int_0^{4(1-x^2-y^2)} dz\\ &= 4\int_0^1 dx\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\\ &= 4\int_0^1\left[\left(y-x^2y-\frac{y^3}{3}\right)\Big|_{y=0}^{\sqrt{1-x^2}}\right]dx = \frac{8}{3}\int_0^1 (1-x^2)^{3/2}\,dx.\end{aligned}$$

The change of variable $x = \sin\theta$ yields

$$V = \frac{8}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2} (1+\cos 2\theta)^2\, d\theta$$
$$= \frac{2}{3}\int_0^{\pi/2} (1+\cos 2\theta + \cos^2 2\theta)\, d\theta = \frac{2}{3}\int_0^{\pi/2} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta = \frac{\pi}{2}.$$

7.2.19. **(a)**

$$\begin{aligned}\int_R \left(\sum_{j=1}^n x_j\right) d\mathbf{X} &= \sum_{j=1}^n \int_R x_j\, d\mathbf{X} \\ &= (a_1-b_1)(a_2-b_2)\cdots(a_n-b_n)\sum_{j=1}^n \frac{1}{b_j-a_j}\int_{a_j}^{b_j} x_j\, dx_j \\ &= (a_1-b_1)(a_2-b_2)\cdots(a_n-b_n)\sum_{j=1}^n \left.\frac{x_j^2}{2(b_j-a_j)}\right|_{a_j}^{b_j} \\ &= \frac{(a_1-b_1)(a_2-b_2)\cdots(a_n-b_n)}{2}\sum_{j=1}^n (a_j+b_j).\end{aligned}$$

(b)

$$\begin{aligned}\int_R \left(\sum_{j=1}^n x_j^2\right) d\mathbf{X} &= \sum_{j=1}^n \int_R x_j\, d\mathbf{X} \\ &= (a_1-b_1)(a_2-b_2)\cdots(a_n-b_n)\sum_{j=1}^n \frac{1}{b_j-a_j}\int_{a_j}^{b_j} x_j^2\, dx_j \\ &= (a_1-b_1)(a_2-b_2)\cdots(a_n-b_n)\sum_{j=1}^n \left.\frac{x_j^3}{3(b_j-a_j)}\right|_{a_j}^{b_j} \\ &= \frac{(a_1-b_1)(a_2-b_2)\cdots(a_n-b_n)}{3}\sum_{j=1}^n (a_j^2+a_jb_j+b_j^2).\end{aligned}$$

(c)

$$\begin{aligned}\int_R x_1x_2\cdots x_n\, d\mathbf{X} &= \left(\int_{a_1}^{b_1} x_1\, dx_1\right)\left(\int_{a_2}^{b_2} x_2\, dx_2\right)\cdots\left(\int_{a_n}^{b_n} x_n\, dx_n\right) \\ &= 2^{-n}(b_1^2-a_1^2)(b_2^2-a_2^2)\cdots(b_n^2-a_n^2).\end{aligned}$$

7.2.21. $S = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned}
S_1 &= \left\{(x, y) \,\middle|\, -1 \le x \le 1,\ 0 \le y \le 1\right\},\\
S_2 &= \left\{(x, y) \,\middle|\, 0 \le x \le y-1,\ 1 \le y \le 2\right\},\\
S_3 &= \left\{(x, y) \,\middle|\, -1 \le x \le 1-y,\ 1 \le y \le 2\right\}.
\end{aligned}$$

$$\begin{aligned}
\int_{S_1} (x+y)\,d(x,y) &= \int_0^1 dy \int_{-1}^1 (x+y)\,dx\\
&= \int_0^1 \left[\left(\frac{x^2}{2} + xy\right)\Bigg|_{x=-1}^{1}\right] dy = 2\int_0^1 y\,dy = 1;
\end{aligned}$$

$$\begin{aligned}
\int_{S_2} (x+y)\,d(x,y) &= \int_1^2 dy \int_0^{y-1} (x+y)\,dx = \int_1^2 \left[\left(\frac{x^2}{2} + xy\right)\Bigg|_{x=0}^{y-1}\right] dy\\
&= \int_1^2 \left(\frac{3y^2}{2} - 2y + \frac{1}{2}\right) dy = \left(\frac{y^3}{2} - y^2 + \frac{y}{2}\right)\Bigg|_1^2 = 1;
\end{aligned}$$

$$\begin{aligned}
\int_{S_3} (x+y)\,d(x,y) &= \int_1^2 dy \int_{-1}^{1-y} (x+y)\,dx = \int_1^2 \left[\left(\frac{x^2}{2} + xy\right)\Bigg|_{x=-1}^{1-y}\right] dy\\
&= \int_1^2 \left(y - \frac{y^2}{2}\right) dy = \left(\frac{y^2}{2} - \frac{y^3}{6}\right)\Bigg|_1^2 = \frac{1}{3};
\end{aligned}$$

therefore, $\displaystyle\int_S (x+y)\,d(x,y) = 1 + 1 + \frac{1}{3} = \frac{7}{3}$.

7.2.22. Reversing the order of integration yields

$$\begin{aligned}
\int_0^1 x\,dx \int_0^{\sqrt{1-x^2}} \frac{dy}{\sqrt{x^2+y^2}} &= \int_0^1 dy \int_0^{\sqrt{1-y^2}} \frac{x\,dx}{\sqrt{x^2+y^2}}\\
&= \int_0^1 \left[\left(\sqrt{x^2+y^2}\right)\Bigg|_{x=0}^{\sqrt{1-y^2}}\right] dy\\
&= \int_0^1 (1-y)\,dy = \left(y - \frac{y^2}{2}\right)\Bigg|_0^1 = \frac{1}{2}.
\end{aligned}$$

7.2.23. Integrating $y^{(n)}(x) = f(x)$ from a to x yields (B) $y^{(n-1)}(x) = \displaystyle\int_a^x f(t_1)\,dt_1$, since $y^{(n-1)}(a) = 0$. Integrating (B) from a to x yields $y^{(n-2)}(x) = \displaystyle\int_a^t dt_2 \int_a^{t_2} f(t_1)\,dt_1$, since $y^{(n-2)}(a) = 0$. Continuing in this way yields (A).

(a) Reversing the order of the integrations with respect to t_1 and t_2 as in Example 7.2.11 yields $\int_a^x dt_2 \int_a^{t_2} f(t_1)\,dt_1 = \int_a^x (x-t_1)f(t_1)\,dt_1$. Now complete the proof by induction. If $n > 2$ and

$$\int_a^x dt_{n-1} \int_a^{t_{n-1}} dt_{n-2} \cdots \int_a^{t_3} dt_2 \int_a^{t_2} f(t_1)\,dt_1 = \frac{1}{(n-2)!}\int_a^x (x-t_1)^{n-2} f(t_1)\,dt_1,$$

then

$$\int_a^x dt_n \int_a^{t_n} dt_{n-1} \cdots \int_a^{t_3} dt_2 \int_a^{t_2} f(t_1)\,dt_1 = \frac{1}{(n-2)!}\int_a^x dt_n \int_a^{t_n} (t_n-t_1)^{n-2} f(t_1)\,dt_1,$$

and reversing the order of the integrations with respect to t_1 and t_n on the right as in Example 7.2.11 yields

$$\int_a^x dt_n \int_a^{t_n} dt_{n-1} \cdots \int_a^{t_3} dt_2 \int_a^{t_2} f(t_1)\,dt_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt,$$

where we have changed the dummy variable t_1 to t.

7.2.24.

$$\begin{aligned}\int_{T_\rho} e^{-xy} \sin ax\, d(x,y) &= \int_0^\rho dy \int_0^\rho e^{-xy} \sin ax\, dx \\ &= \int_0^\rho \frac{a}{a^2+y^2}\,dy - \int_0^\rho \frac{e^{-\rho y}(y\sin a\rho + a\cos a\rho)}{a^2+\rho^2}\,dy.\end{aligned}$$

By Schwarz's inequality,

$$\left|\frac{y\sin a\rho + a\cos a\rho}{a^2+\rho^2}\right| \le \frac{1}{\sqrt{a^2+\rho^2}},$$

so the second integral is less in magnitude than $\frac{1}{a}\int_0^\infty e^{-\rho y}\,dy = \frac{1}{\rho a} \to 0$ as $\rho \to \infty$; hence,

$$I(a) = \int_0^\infty \frac{a}{a^2+y^2}\,dy = \tan^{-1}\frac{y}{a}\Big|_0^\infty = \frac{\pi}{2}. \qquad \text{(A)}$$

Reversing the order of integration yields

$$\begin{aligned}\int_{T_\rho} e^{-xy} \sin ax\, dx &= \int_0^\rho \sin ax\, dx \int_0^\rho e^{-xy}\,dy \\ &= \int_0^\rho \frac{1-e^{-x\rho}}{x} \sin ax\, dx \\ &= \int_0^\rho \frac{\sin ax}{x} dx - \int_0^\rho e^{-x\rho}\frac{\sin ax}{x} dx.\end{aligned}$$

But $\left|e^{-xp}\dfrac{\sin ax}{x}\right| \le |a|e^{-xp}$, so the second integral $\to 0$ as $p \to \infty$. Hence, $I(a) = \displaystyle\int_0^\infty \frac{\sin ax}{x}dx$. This and (A) yield the conclusion.

7.3 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

7.3.1. Let S_1 and S_2 be dense in R and $S_1 \cup S_2 = R$; for example, let S_1 be the set of points in R with rational coordinates and let S_2 be the set of points in R with at least one irrational coordinate. Then $\overline{\int_R} \psi_{S_1 \cup S_2}\, d\mathbf{X} = \overline{\int_R} \psi_{S_1 \cup S_2}\, d\mathbf{X} = V(R)$, while $\overline{\int_R} \psi_{S_1}\, d\mathbf{X} = \overline{\int_R} \psi_{S_2}\, d\mathbf{X} = V(R)$, so $\overline{\int_R} \psi_{S_1}\, d\mathbf{X} + \overline{\int_R} \psi_{S_2}\, d\mathbf{X} = 2V(R)$, and $\underline{\int_R} \psi_{S_1}\, d\mathbf{X} = \underline{\int_R} \psi_{S_2}\, d\mathbf{X} = 0$, so $\underline{\int_R} \psi_{S_1}\, d\mathbf{X} + \underline{\int_R} \psi_{S_2}\, d\mathbf{X} = 0$,

7.3.2. Let $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be a rectangle containing E and let $\epsilon > 0$. Suppose that E has Jordan content zero. Since $\displaystyle\int_R \psi_E(\mathbf{X})\, d\mathbf{X} = 0$, there is partition $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ of R such that $S_{\psi_E}(P) < \epsilon$; that is, if $U = \left\{ j \,\middle|\, R_j \cap E \ne \emptyset \right\}$, then $\displaystyle\sum_{j \in U} V(T_j) < \epsilon$. Since $E \subset \displaystyle\bigcup_{j \in U} T_j$, E has content zero in the sense of Definition 7.1.14.

Now suppose that E has content zero in the sense of Definition 7.1.14. Then there are rectangles $T_j = [a_{1j}, b_{1j}] \times [a_{2j}, b_{2j}] \times \cdots [a_{nj}, b_{nj}]$ $(1 \le j \le n)$ in R such that $E \subset \displaystyle\bigcup_{j=1}^m T_j$ and $\displaystyle\sum_{j=1}^m V(T_j) < \epsilon$. For $1 \le i \le n$ let P_i be a partition of $[a_i, b_i]$ that includes the partition points $a_i, b_i, a_{i1}, b_{i1}, a_{i2}, b_{i2}, \dots, a_{in}, b_{in}$, and let $\mathbf{P} = P_1 \times P_2 \times \cdots \times P_n$. Then (A) $S_{\psi_E}(\mathbf{P}) < \epsilon$. Since $s_{\psi_E} > 0$, it follows that $\displaystyle\int_R \psi_E(\mathbf{X})\, d\mathbf{X}$ exists (Theorem 7.1.12) and equals $\overline{\int_R} \psi_E(\mathbf{X})\, d\mathbf{X}$ (Theorem 7.1.8). Since $0 \le \overline{\int_R} \psi_E(\mathbf{X})\, d\mathbf{X} \le S_{\psi_E}(\mathbf{P})$, (A) implies that $\overline{\int_R} \psi_E(\mathbf{X})\, d\mathbf{X} = 0$; that is, E has Jordan content zero.

7.3.3. Since S_1 and S_2 are Jordan measurable, ∂S_1 and ∂S_2 have zero content (Theorem 7.3.1). Therefore, $\partial S_1 \cup \partial S_2$ has zero content. Since $\partial(S_1 \cup S_2) \subset \partial S_1 \cup \partial S_2$ and $\partial(S_1 \cap S_2) \subset \partial S_1 \cup \partial S_2$ (Exercise1.3.24**(a)**,**(b)**), $\partial(S_1 \cup S_2)$ and $\partial(S_1 \cap S_2)$ have zero content. Therefore, $S_1 \cup S_2$ and $S_1 \cap S_2$ are Jordan measurable (Theorem 7.3.1).

7.3.4. **(a)** Since S is Jordan measurable, S is bounded (by definition). Therefore, ∂S has zero content (Theorem 7.3.1). Since $\partial\overline{S} \subset \partial S$ (Exercise 1.3.24**(c)**), $\partial\overline{S}$ has zero content. Therefore, $\partial\overline{S}$ is Jordan measurable (Theorem 7.3.1).

No; $S = \left\{(x, y) \,\middle|\, 0 \le x \le 1,\ 0 \le y \le 1,\ x, y \text{ rational}\right\}$ is not Jordan measurable, but $\overline{S}$ is.

(b) Since S and T are both Jordan measurable, ∂S and ∂T have zero content (Theorem 7.3.1). Since $\partial(S - T) \subset \partial S \cup \partial T$ (Exercise 1.3.24**(c)**), $\partial(S - T)$ has zero content. Therefore, $S - T$ is Jordan measurable (Theorem 7.3.1).

7.3.5. Suppose that $\epsilon > 0$. From Theorem 7.3.1, $V(\partial S) = 0$, so ∂S can be covered by rectangles $R_1, \dots, R_k$ such that (A) $\displaystyle\sum_{i=1}^{k} V(R_i) < \epsilon$; moreover, we may assume that $\partial S \subset T = \displaystyle\bigcup_{i=1}^{k} R^0$. Now $S \cap T^c$ is a closed subset of S^0, and

$$H = [H \cap (S \cap T^c)] \cup [H \cap T]. \tag{B}$$

But $V\,[H \cap (S \cap T^c)] = 0$ by assumption, so $H \cap (S \cap T^c)$ can be covered by finitely many rectangles with total content $< \epsilon$. Now (A) and (B) and the definition of T imply that H can be covered by a collection of rectangles with total content $< 2\epsilon$. Hence, $V(H) = 0$.

7.3.6. Define $\delta_{rs} = \begin{cases} 1 & \text{if } r = s, \\ 0 & \text{if } r \ne s, \end{cases}$ so $\mathbf{I} = [\delta_{rs}]$. Let $\mathbf{A} = [a_{rs}]$, $\mathbf{E} = [e_{rs}]$, and $\mathbf{EA} = [b_{rs}]$. In all cases, $b_{rs} = \displaystyle\sum_{k=1}^{n} e_{rk} a_{kr}$.

(a) If $\mathbf{E}$ is obtained by interchanging rows i and j of $\mathbf{I}$, then

$$e_{rs} = \begin{cases} \delta_{rs} & \text{if } r \ne i \text{ and } r \ne j, \\ \delta_{js} & \text{if } r = i, \\ \delta_{is} & \text{if } r = j. \end{cases}$$

Therefore, $b_{rs} = \begin{cases} \delta_{rr} a_{rs} = a_{rs} & , \text{ if } r \ne i \text{ and } r \ne s, \\ \delta_{jj} a_{js} = a_{js} & , \text{ if } r = i, \\ \delta_{ii} a_{is} = a_{is} & , \text{ if } r = j. \end{cases}$

(b) If $\mathbf{E}$ is obtained by multiplying row i of $\mathbf{I}$ by a constant c, then

$$e_{rs} = \begin{cases} \delta_{rs} & \text{if } r \ne i, \\ c\delta_{is} & \text{if } r = i. \end{cases}$$

Therefore, $b_{rs} = \begin{cases} \delta_{rr} a_{rs} = a_{rs} & , \text{ if } r \ne i, \\ c\delta_{ii} a_{is} = c a_{is} & , \text{ if } r = i. \end{cases}$

(b) If $\mathbf{E}$ is obtained by adding c times row i of $\mathbf{I}$ to row j $(j \ne i)$, then

$$e_{rs} = \begin{cases} \delta_{rs} & \text{if } r \ne j, \\ \delta_{js} + c\delta_{is} & \text{if } r = j. \end{cases}$$

Therefore, $b_{rs} = \begin{cases} \delta_{rr} a_{rs} = a_{rs} & , \text{ if } r \ne j, \\ \delta_{jj} a_{js} + c\delta_{ii} a_{is} = a_{js} + c a_{is} & , \text{ if } r = j. \end{cases}$

7.3.7. **(a)** Note that det(**I**) $= 1$. Interchanging two rows of a matrix multiplies its determinant by -1; hence, det(**E**) $= -1$ if **E** is of type **(a)**. Multiplying a row of a matrix by a constant c multiplies its determinant by c; hence, det(**E**) $= c$ if **E** is of type **(b)**.

Adding a multiple of one row of a matrix to another does not change its determinant; hence, $\det(\mathbf{E}) = 1$ if $\mathbf{E}$ is of type **(c)**.

(b) We must have $\mathbf{E}^{-1}\mathbf{E} = \mathbf{I}$ in all cases. If $\mathbf{E}$ interchanges rows i and j, then $\mathbf{E}^{-1} = \mathbf{E}$. If $\mathbf{E}$ multiplies row i by c, then $\mathbf{E}^{-1}$ multiplies row i by $1/c$. If $\mathbf{E}$ adds c times row i to row j then $\mathbf{E}^{-1}$ adds $-c$ times row i to row j.

7.3.8. **(a)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$; $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \mathbf{E}_1\mathbf{A}$ with $\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{E}_2\mathbf{A}_1 \quad \text{with} \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix};$$

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_3\mathbf{A}_2 \quad \text{with} \quad \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix};$$

$$\mathbf{A}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_4\mathbf{A}_3 \quad \text{with} \quad \mathbf{E}_4 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{I} = \mathbf{E}_5\mathbf{A}_4 \quad \text{with} \quad \mathbf{E}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I} = (\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)\mathbf{A};$$

$$\begin{aligned} \mathbf{A} &= \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1}\mathbf{E}_5^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 3 & -2 \\ 0 & -1 & 5 \\ 0 & -2 & 4 \end{bmatrix}$; $\mathbf{A}_1 = \begin{bmatrix} 1 & \frac{3}{2} & -1 \\ 0 & -1 & 5 \\ 0 & -2 & 4 \end{bmatrix} = \mathbf{E}_1\mathbf{A}$ with $\mathbf{E}_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\mathbf{A}_2 = \begin{bmatrix} 1 & \frac{3}{2} & -1 \\ 0 & 1 & -5 \\ 0 & -2 & 4 \end{bmatrix} = \mathbf{E}_2\mathbf{A}_1 \quad \text{with} \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{A}_3 = \begin{bmatrix} 1 & \frac{3}{2} & -1 \\ 0 & 1 & -5 \\ 0 & 0 & -6 \end{bmatrix} = \mathbf{E}_3\mathbf{A}_2 \quad \text{with} \quad \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix};$$

$$\mathbf{A}_4=\begin{bmatrix}1&\frac{3}{2}&-1\\0&1&-5\\0&0&1\end{bmatrix}=\mathbf{E}_4\mathbf{A}_3\quad\text{with}\quad\mathbf{E}_4=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{6}\end{bmatrix};$$

$$\mathbf{A}_5=\begin{bmatrix}1&0&\frac{13}{2}\\0&1&-5\\0&0&1\end{bmatrix}=\mathbf{E}_5\mathbf{A}_4\quad\text{with}\quad\mathbf{E}_5=\begin{bmatrix}1&-\frac{3}{2}&0\\0&1&0\\0&0&1\end{bmatrix};$$

$$\mathbf{A}_6=\begin{bmatrix}1&0&0\\0&1&-5\\0&0&1\end{bmatrix}=\mathbf{E}_6\mathbf{A}_5\quad\text{with}\quad\mathbf{E}_6=\begin{bmatrix}1&0&-\frac{13}{2}\\0&1&0\\0&0&1\end{bmatrix};$$

$$\mathbf{I}=\mathbf{E}_7\mathbf{A}_6\quad\text{with}\quad\mathbf{E}_7=\begin{bmatrix}1&0&0\\0&1&5\\0&0&1\end{bmatrix};\quad\mathbf{I}=(\mathbf{E}_7\mathbf{E}_6\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)\mathbf{A};$$

$$\begin{aligned}\mathbf{A}&=\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1}\mathbf{E}_5^{-1}\mathbf{E}_6^{-1}\mathbf{E}_7^{-1}\\&=\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-6\end{bmatrix}\\&\quad\times\begin{bmatrix}1&\frac{3}{2}&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&\frac{13}{2}\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-5\\0&0&1\end{bmatrix}.\end{aligned}$$

7.3.9. $\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{A}\begin{bmatrix}u\\v\end{bmatrix}$ with $\mathbf{A}=\dfrac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$. Since $\det(\mathbf{A})=\dfrac{1}{ad-bc}$, Theorem 7.3.7 with $S=[u_1,u_2]\times[v_1,v_2]$ implies that $V(\mathbf{L}(S))=\dfrac{(u_2-u_1)(v_2-v_1)}{|ad-bc|}$.

7.3.10. $\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{A}\begin{bmatrix}u\\v\\w\end{bmatrix}$ with $\mathbf{A}=\begin{bmatrix}2&3&-2\\-1&5&0\\-2&4&0\end{bmatrix}^{-1}$. Since $\det(\mathbf{A})=-\dfrac{1}{12}$, Theorem 7.3.7 with $S=[1,2]\times[5,7]\times[1,6]$ implies that $V=\dfrac{5}{6}$.

7.3.11. If $\mathbf{X}\in H=\mathbf{G}(S_1)\cap\mathbf{G}(S\cap S_1^c)$, then $\mathbf{X}=\mathbf{G}(\mathbf{U}_1)$ and $\mathbf{X}=\mathbf{G}(\mathbf{U}_2)$ where $\mathbf{U}_1\in S_1$ and $\mathbf{U}_2\in S\cap S_1^c$. Since $\mathbf{G}$ is one-to-one on S^0 and $\mathbf{U}_1\in S^0$, it follows that $\mathbf{U}_2\in\partial S$. Therefore, $H\subset\mathbf{G}(\partial S)$, so $V(H)=0$, because $V(\mathbf{G}(\partial S))=0$, by Theorem 7.3.1 and Lemma 7.3.4. Now use Corollary 7.1.31.

7.3.12. We show that $I_2=I_1$. Similar arguments show that $I_3=I_1$ and $I_4=I_1$. Note that

$$\begin{aligned}T_1&=\left\{(x,y)\,\middle|\,a^2\le x^2+y^2\le b^2,\ x\ge 0,\ y\ge 0\right\},\\T_2&=\left\{(u,v)\,\middle|\,a^2\le u^2+v^2\le b^2,\ u\le 0,\ v\ge 0\right\}\end{aligned}$$

Consider the transformation $(x,y)=\mathbf{G}(u,v)=(-u,v)$. Then $\mathbf{G}(T_2)=T_1$. Since $J\mathbf{G}(u,v)=-1$, Theorem 7.3.8 with $S=T_2$ implies that $\displaystyle\int_{T_1}f(x,y)\,d(x,y)=\int_{T_2}f(-u,v)\,d(u,v)$, which can be rewritten as (A) $\int_{T_1} f(x,y)\,d(x,y) = \int_{T_2} f(-x,y)\,d(x,y)$, since the names of the variables of integration are irrelevant. Since $f(-x,y) = f(x,y)$, (A) implies that $I_2 = I_1$.

7.3.13. **(a)** If $\mathbf{X} = (x_1, x_2, \dots, x_n)$ let $\mathbf{H}(\mathbf{X}) = (e_1x_1, e_2x_2, \dots, e_nx_n)$ and, if U is any set, let $\widehat{U} = \left\{\mathbf{H}(\mathbf{X}) \,\middle|\, \mathbf{X} \in U\right\}$. If R is a rectangle containing T, then $\widehat{R}$ is a rectangle containing $\widehat{T}$. If $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ is a partition of R then $\widehat{\mathbf{P}} = \{\widehat{R}_1, \widehat{R}_2, \dots \widehat{R}_k\}$ is a partition of $\widehat{R}$. If $\sigma = \sum_{j=1}^{k} f(\mathbf{X}_j)V(R_j)$ is a Riemann sum of f_T over R, then $\widehat{\sigma} = e_0 \sum_{j=1}^{k} g(\mathbf{H}(\mathbf{X}_j))V(\widehat{R}_j)$ is a Riemann sum of $g_{\widehat{T}}$ over $\widehat{R}$, and conversely. However, since $g(\mathbf{H}(\mathbf{X}_j)) = f(\mathbf{X}_j)$ and $V(\widehat{R}_j) = V(R_j)$, $\widehat{\sigma} = e_0\sigma$. This implies the conclusion.

(b) Define $g(\mathbf{H}(\mathbf{X})) = f(\mathbf{X})$; that is, let $e_0 = 1$. From **(a)**, g is integrable on $\widehat{T} = T$ and $\int_T g(\mathbf{Y})\,d\mathbf{Y} = \int_T f(\mathbf{X})\,d\mathbf{X}$, which is equivalent to (A) $\int_T g(\mathbf{X})\,d\mathbf{X} = \int_T f(\mathbf{X})\,d\mathbf{X}$, since the name of the variable of integration is irrelevant. Since $\mathbf{H}(\mathbf{H}(\mathbf{X})) = \mathbf{X}$, $g(\mathbf{X}) = f(\mathbf{H}(\mathbf{X})) = -f(\mathbf{X})$. Therefore, (A) implies that $-\int_T f(\mathbf{X})\,d\mathbf{X} = \int_T f(\mathbf{X})\,d\mathbf{X}$, so $\int_T f(\mathbf{X})\,d\mathbf{X} = 0$.

7.3.14. **(a)** Let $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{F}(x,y) = \begin{bmatrix} y/x \\ x+2y \end{bmatrix}$; then $\mathbf{F}'(x,y) = \begin{bmatrix} -y/x^2 & 1/x \\ 1 & 2 \end{bmatrix}$, so

$$J\mathbf{F}(x,y) = \frac{2y+x}{x^2} = -\frac{(2y+x)^2}{x^2}\frac{1}{2y+x} = \frac{(1+2u)^2}{v}.$$

If $\mathbf{G} = \mathbf{F}^{-1}$ then $J\mathbf{G}(u,v) = -\dfrac{v}{(1+2u)^2}$, so Theorem 7.3.15 with $f = 1$ yields $V = \int_S \dfrac{v}{(1+2u)^2}\,d(u,v)$, with $S = [1,4] \times [1,3]$. Therefore,

$$V = \int_1^4 \frac{du}{(1+2u)^2} \int_1^3 v\,dv = \left[-\frac{1}{2(1+2u)}\bigg|_1^4\right]\left[\frac{v^2}{2}\bigg|_1^3\right] = \left(\frac{1}{9}\right)4 = \frac{4}{9}.$$

(b) Let $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{F}(x,y) = \begin{bmatrix} xy \\ y/x \end{bmatrix}$; then $\mathbf{F}'(x,y) = \begin{bmatrix} y & x \\ -y/x^2 & 1/x \end{bmatrix}$ and $J\mathbf{F}(x,y) = \dfrac{2y}{x} = 2v$. If $\mathbf{G} = \mathbf{F}^{-1}$ then $J\mathbf{G}(u,v) = \dfrac{1}{2v}$, so Theorem 7.3.15 with $f = 1$ yields $V = \int_S \dfrac{1}{2v}\,d(u,v)$, with $S = [2,4] \times [2,5]$. Therefore, $V = \dfrac{1}{2}\int_2^4 du \int_2^5 \dfrac{dv}{v} = \log\dfrac{5}{2}$.

7.3.15. Let

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \text{then} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{G}(u, v, w),$$

where $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ and $J\mathbf{G}(u, v, w) = \det(\mathbf{A}) = \frac{1}{2}$. From Theorem 7.3.15,

$$\begin{aligned} \int_T (3x^2 + 2y + z) d(x, y, z) &= \frac{1}{2} \int_S \left[3 \left(\frac{u + v + w}{2} \right)^2 + 2 \left(\frac{-u + v}{2} \right) + \left(\frac{-u - v + w}{2} \right) \right] d(u, v, w) \\ &= \frac{1}{8} \int_S (3u^2 + 6uv + 6uw - 6u + 3v^2 + 6vw + 2v + 3w^2 + 2w)\, d(u, v, w) \end{aligned}$$

where $S = [-1, 1] \times [-1, 1] \times [-1, 1]$. Exploiting the symmetries in S reduces this to

$$\int_T (3x^2 + 2y + z) d(x, y, z) = \frac{9}{8} \int_{-1}^{1} du \int_{-1}^{1} dv \int_{-1}^{1} w^2\, dw = 3.$$

7.3.16. Let $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{F}(x, y) = \begin{bmatrix} xy \\ y - x^2 \end{bmatrix}$; then $\mathbf{F}'(x, y) = \begin{bmatrix} y & x \\ -2x & 1 \end{bmatrix}$ and $J\mathbf{F}(x, y) = y + 2x^2$. Theorem 7.3.15 with $\mathbf{G} = \mathbf{F}^{-1}$ implies that

$$\int_T (y^2 + x^2 y - 2x^4)\, d(x, y) = \int_S \frac{y^2 + x^2 y - 2x^4}{y + 2x^2}\, d(u, v) = \int_S (y - x^2)\, d(u, v) = \int_S v\, d(u, v),$$

where $S = [1, 2] \times [0, 1]$. Therefore, $\int_T (y^2 + x^2 y - 2x^4)\, d(x, y) = \int_1^2 du \int_0^1 v\, dv = \frac{1}{2}$.

7.3.17. Let $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{F}(x, y) = \begin{bmatrix} xy \\ x^2 - y^2 \end{bmatrix}$; then $\mathbf{F}'(x, y) = \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix}$ and $J\mathbf{F}(x, y) = -2(x^2 + y^2)$. Theorem 7.3.15 with $\mathbf{G} = \mathbf{F}^{-1}$ implies that

$$\int_T (x^4 - y^4) e^{xy}\, d(x, y) = \frac{1}{2} \int_S \frac{x^4 - y^4}{x^2 + y^2} e^{xy}\, d(u, v) = \frac{1}{2} \int_S (x^2 - y^2) e^{xy}\, d(u, v) = \frac{1}{2} \int_S v e^u\, d(u, v),$$

with $S = [1, 2] \times [2, 3]$. Therefore, $V = \frac{1}{2} \int_1^2 e^u\, du \int_2^3 v\, dv = \frac{5}{4} e(e - 1)$.

7.3.18. Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{G}(r, \theta, \phi) = \begin{bmatrix} ar\cos\theta\cos\phi \\ br\sin\theta\cos\phi \\ cr\sin\phi \end{bmatrix};$$

then

$$G'(r, \theta, \phi) = \begin{bmatrix} a\cos\theta\cos\phi & -ar\sin\theta\cos\phi & -ar\cos\theta\sin\phi \\ b\sin\theta\cos\phi & br\cos\theta\cos\phi & -br\sin\theta\sin\phi \\ c\sin\phi & 0 & cr\cos\phi \end{bmatrix}$$

and $JG(r,\theta,\phi) = abc\,r^2\cos\phi$. Theorem 7.3.15 implies that $V = abc\int_S r^2\cos\phi\, d(r,\theta,\phi)$ with $S = [0,1]\times[0,2\pi]\times\left[\frac{\pi}{2}, -\frac{\pi}{2}\right]$. Therefore,

$$V = abc\int_0^1 r^2\,dr\int_0^{2\pi} d\theta\int_{-\pi/2}^{\pi/2}\cos\phi\,d\phi = \frac{4}{3}\pi abc.$$

7.3.19. From the discussion of spherical coordinates,

$$\int_T \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\,d(x,y,z) = \int_S \frac{e^{r^2}}{r} r^2\cos\phi\,d(r,\theta,\phi) = \int_S re^{r^2}\cos\phi\,d(r,\theta,\phi)$$

with $S = [3,5]\times[0,2\pi]\times\left[\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore,

$$\int_T \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\,d(x,y,z) = \int_3^5 re^{r^2}\,dr\int_0^{2\pi} d\theta\int_{-\pi/2}^{\pi/2}\cos\phi\,d\phi = 2\pi(e^{25}-e^9).$$

7.3.20. Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{G}(r,\theta,z) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$; then $\mathbf{G}'(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $J\mathbf{G}(r,\theta,z) = r$. Theorem 7.3.15 implies that $V = \int_S r\,d(r,\theta,z)$ with $S = \left\{(r,\theta,z)\,\middle|\, 0\le z\le r,\ 0\le r\le 2,\ 0\le\theta\le 2\pi\right\}$. Therefore, $V = \int_0^{2\pi} d\theta\int_0^2 r\,dr\int_0^z dz = 2\pi\int_0^2 r^2\,dr = \frac{16\pi}{3}$.

7.3.21. Let $\begin{bmatrix} u \\ v \\ z \end{bmatrix} = \mathbf{F}(x,y,z) = \begin{bmatrix} x^2-y^2 \\ x^2+y^2 \\ z \end{bmatrix}$; then $\mathbf{F}'(x,y,z) = \begin{bmatrix} 2x & -2y & 0 \\ 2x & 2y & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $J\mathbf{F}(x,y,z) = 8xy$. If $\mathbf{G} = \mathbf{F}^{-1}$, then $J\mathbf{G}(u,v,z) = \frac{1}{8xy}$, and Theorem 7.3.15 with implies that $\int_T xyz(x^4-y^4)d(x,y,z) = \frac{1}{8}\int_S uvz\,d(u,v,z)$, where $S = [1,2]\times[3,4]\times[0,1]$. Therefore, $\int_T xyz(x^4-y^4)d(x,y,z) = \frac{1}{8}\int_1^2 u\,du\int_3^4 v\,dv\int_0^1 z\,dz = \frac{21}{64}$.

7.3.22. In all parts denote the iterated integral by I.

(a) $I = \int_T \frac{d(x,y)}{1+x^2+y^2}$, where $T = \left\{(x,y)\,\middle|\, y\le x\le\sqrt{4-y^2},\ 0\le y\le\sqrt{2}\right\}$, which is the image of $S = \left\{(r,\theta)\,\middle|\, 0\le r\le 2,\ 0\le\theta\le\pi/4\right\}$ under the transformation $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{G}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$. Since $J\mathbf{G}(r,\theta) = r$, Theorem 7.3.15 implies that $I = \int_0^{\pi/4} d\theta\int_0^2 \frac{r\,dr}{1+r^2} = \frac{\pi}{8}\log 5$.

(b) $I = \int_T e^{x^2+y^2}\,d(x,y)$, where $T = \left\{(x,y)\,\middle|\,0 \le y \le \sqrt{4-x^2},\ 0 \le x \le 2\right\}$, which is the image of $S = \left\{(r,\theta)\,\middle|\,0 \le r \le 2,\ 0 \le \theta \le \pi/2\right\}$ under the transformation $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{G}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$. Since $J\mathbf{G}(r,\theta) = r$, Theorem 7.3.15 implies that $I = \int_S re^{r^2}\,d(r,\theta) = \int_0^{\pi/2} d\theta \int_0^2 re^{r^2}\,dr = \frac{\pi}{4}(e^4 - 1)$.

(c) $I = \int_T z^2\,d(x,y,z)$ where

$$T = \left\{(x,y,z)\,\middle|\,0 \le z \le \sqrt{1-x^2-y^2},\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ -1 \le x \le 1\right\},$$

which is the image of $S = \left\{(r,\theta,\phi)\,\middle|\,0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2\right\}$ under the transformation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{G}(r,\theta) = \begin{bmatrix} r\cos\theta\cos\phi \\ r\sin\theta\cos\phi \\ r\cos\phi \end{bmatrix}$. Since $J\mathbf{G}(r,\theta) = r^2\cos\phi$, Theorem 7.3.15 implies that $I = \int_S r^4\cos\phi\sin^2\phi\,d(r,\theta,\phi) = \int_0^{2\pi} d\theta \int_0^1 r^4\,dr \int_0^{\pi/2} \cos\phi\sin^2\phi\,d\phi = \frac{2\pi}{15}$.

7.3.23. By symmetry, the 4-ball is the union of 16 sets, each of which has the same volume as

$$T_1 = \left\{(x_1,x_2,x_3,x_4)\,\middle|\,x_1^2 + x_2^2 + x_3^2 + x_4^2 \le a^2,\ x_1,x_2,x_3,x_4 \ge 0\right\}.$$

Moreover, $T_1 = \mathbf{G}(S_1)$, where

$$S_1 = \left\{(r,\theta_1,\theta_2,\theta_3)\,\middle|\,0 \le r \le a,\ 0 \le \theta_1,\theta_2,\theta_3 \le \pi/2\right\},$$

and $\mathbf{G}$ is one-to-one on S_1.

$\mathbf{G}'(r,\theta_1,\theta_2,\theta_3) =$

$$\begin{bmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 & -r\sin\theta_1\cos\theta_2\cos\theta_3 & -r\cos\theta_1\sin\theta_2\cos\theta_3 & -r\cos\theta_1\cos\theta_2\sin\theta_3 \\ \sin\theta_1\cos\theta_2\cos\theta_3 & r\cos\theta_1\cos\theta_2\cos\theta_3 & -r\sin\theta_1\sin\theta_2\cos\theta_3 & -r\sin\theta_1\cos\theta_2\sin\theta_3 \\ \sin\theta_2\cos\theta_3 & 0 & r\cos\theta_2\cos\theta_3 & -r\sin\theta_2\sin\theta_3 \\ \sin\theta_3 & 0 & 0 & r\cos\theta_3 \end{bmatrix}$$

$J\mathbf{G}(r,\theta_1,\theta_2,\theta_3) =$

$$r^3\begin{vmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1\cos\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_2\cos\theta_3 & -\cos\theta_1\cos\theta_2\sin\theta_3 \\ \sin\theta_1\cos\theta_2\cos\theta_3 & \cos\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2\cos\theta_3 & -\sin\theta_1\cos\theta_2\sin\theta_3 \\ \sin\theta_2\cos\theta_3 & 0 & \cos\theta_2\cos\theta_3 & -\sin\theta_2\sin\theta_3 \\ \sin\theta_3 & 0 & 0 & \cos\theta_3 \end{vmatrix}$$

$= r^3(-D_1 \sin\theta_3 + D_2 \cos\theta_3$, with

$$\begin{aligned}
D_1 &= \begin{vmatrix} -\sin\theta_1\cos\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_2\cos\theta_3 & -\cos\theta_1\cos\theta_2\sin\theta_3 \\ \cos\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2\cos\theta_3 & -\sin\theta_1\cos\theta_2\sin\theta_3 \\ 0 & \cos\theta_2\cos\theta_3 & -\sin\theta_2\sin\theta_3 \end{vmatrix} \\
&= \cos\theta_2\cos^2\theta_3\sin\theta_3 \begin{vmatrix} -\sin\theta_1 & -\cos\theta_1\sin\theta_2 & -\cos\theta_1\cos\theta_2 \\ \cos\theta_1 & -\sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2 \\ 0 & \cos\theta_2 & -\sin\theta_2 \end{vmatrix} \\
&= -\cos^3\theta_2\cos^2\theta_3\sin\theta_3 \begin{vmatrix} -\sin\theta_1 & -\cos\theta_1 \\ \cos\theta_1 & -\sin\theta_1 \end{vmatrix} \\
&\quad -\cos\theta_2\sin^2\theta_2\cos^2\theta_3\sin\theta_3 \begin{vmatrix} -\sin\theta_1 & -\cos\theta_1 \\ \cos\theta_1 & -\sin\theta_1 \end{vmatrix} \\
&= -\cos\theta_2\cos^2\theta_3\sin\theta_3(\cos^2\theta_2 + \sin^2\theta_2) = -\cos\theta_2\cos^2\theta_3\sin\theta_3.
\end{aligned}$$

and

$$\begin{aligned}
D_2 &= \begin{vmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1\cos\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_2\cos\theta_3 \\ \sin\theta_1\cos\theta_2\cos\theta_3 & \cos\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2\cos\theta_3 \\ \sin\theta_2\cos\theta_3 & 0 & \cos\theta_2\cos\theta_3 \end{vmatrix} \\
&= \cos\theta_2\cos^3\theta_3 \begin{vmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_1 & -\cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1 & -\sin\theta_1\sin\theta_2 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{vmatrix} \\
&= \cos\theta_2\sin^2\theta_2\cos^3\theta_3 \begin{vmatrix} -\sin\theta_1 & -\cos\theta_1 \\ \cos\theta_1 & -\sin\theta_1 \end{vmatrix} \\
&\quad + \cos^3\theta_2\cos^3\theta_3 \begin{vmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{vmatrix} \\
&= \cos^3\theta_3\cos\theta_2(\sin^2\theta_2 + \cos^2\theta_2) = \cos\theta_2\cos^3\theta_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J\mathbf{G}(r,\theta_1,\theta_2,\theta_3) &= r^3(-D_1\sin\theta_3 + D_2\cos\theta_3) \\
&= r^3(\cos\theta_2\cos^2\theta_3\sin^2\theta_3 + \cos\theta_2\cos^4\theta_2) \\
&= r^3\cos\theta_2\cos^2\theta_3.
\end{aligned}$$

Now Theorem 7.3.8 implies that

$$\begin{aligned}
V(T_1) &= \int_{S_1} r^3\cos\theta_2\cos^2\theta_3\, d(r,\theta_1,\theta_2,\theta_3) \\
&= \int_0^{\pi/2} d\theta_1 \int_0^{\pi/2} \cos\theta_2\, d\theta_2 \int_0^{\pi/2} \cos^2\theta_3\, d\theta_3 \int_0^2 r^3\, dr = \frac{\pi^2 a^4}{32},
\end{aligned}$$

so $V(T) = 16V(T_1) = \dfrac{\pi^2 a^4}{2}$.

7.3.24. **(a)** Let $\mathbf{Y} = \mathbf{A}\mathbf{X}$; then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$ and $V(T) = |\det(\mathbf{A}^{-1})|V(S)$, where $S = [\alpha_1, \beta_2] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_n, \beta_n]$; that is, $V(T) = \dfrac{(\beta_1-\alpha_1)(\beta_2-\alpha_2)\cdots(\beta_n-\alpha_n)}{|\det(\mathbf{A})|}$.

(b) Since $x_j = \sum_{i=1}^{n} (\mathbf{A}^{-1})_{ji} y_i$, Theorem 7.3.8 implies that

$$\int_T x_j \, d\mathbf{X} = \frac{1}{|\det(\mathbf{A})|} \sum_{i=1}^{n} (\mathbf{A}^{-1})_{ji} \int_S y_i \, d\mathbf{Y}.$$

Therefore,

$$\int_T \left(\sum_{j=1}^{n} c_j x_j \right) d\mathbf{X} = \frac{1}{|\det(\mathbf{A})|} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (\mathbf{A}^{-1})_{ji} c_j \right) \int_S y_i \, d\mathbf{Y} = \frac{1}{|\det(\mathbf{A})|} \sum_{i=1}^{n} d_i \int_S y_i \, d\mathbf{Y}. \tag{A}$$

However, $\int_S y_i \, d\mathbf{Y} = \dfrac{V(S)}{\beta_i - \alpha_i} \int_{\alpha_i}^{\beta_i} y_i \, dy_i = \dfrac{V(S)}{\beta_i - \alpha_i} \dfrac{\beta_i^2 - \alpha_i^2}{2} = \dfrac{V(S)}{2}(\alpha_i + \beta_i)$. Therefore, (A) and **(a)** imply that $\int_T \left(\sum_{j=1}^{n} c_j x_j \right) d\mathbf{X} = \dfrac{V(T)}{2} \sum_{i=1}^{n} d_i(\alpha_i + \beta_i)$.

7.3.25. The ellipsoid is the image of T under the linear transformation $\mathbf{X} = \mathbf{A}\mathbf{Y}$, where $\mathbf{A}$ is the diagonal matrix with $a_1, a_2, \ldots, a_n$ on the diagonal. Therefore, Theorem 7.3.7 implies that the content of the ellipsoid is $|a_1 a_2 \cdots a_n| V_n$.

CHAPTER 8
METRIC SPACES

8.1 INTRODUCTION TO METRIC SPACES

8.1.1. Clearly, **(a)** implies **(i)** and **(b)** and **(c)** imply **(ii)**. Also, **(b)** and **(c)** imply **(ii)**. For the converse, setting $u = v$ in **(ii)** yields $2\rho(w, v) \geq 0$ for any (w, v), and letting $w = u$ here and invoking **(i)** yields **(a)**. Setting $w = v$ in **(ii)** yields (A) $\rho(u, v) \leq \rho(v, u)$ for all u and v in A. Interchanging u and v in (A) yields (C) $\rho(v, u) \leq \rho(u, v)$ for all u and v in A. Now (B) and (C) imply **(b)**. Since $\rho(w, u) = \rho(u, w)$, **(ii)** implies **(c)**.

8.1.2. $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y)$ and $\rho(u, v) \leq \rho(u, x) + \rho(x, y) + \rho(y, v)$ so $\rho(x, u) + \rho(v, y) \geq \rho(x, y) - \rho(u, v)$ and $\rho(u, x) + \rho(y, v) \geq \rho(u, v) - \rho(x, y)$. Since $\rho(u, x) = \rho(x, u)$ and $\rho(y, v) = \rho(v, y)$, the last two inequalities imply that $|\rho(x, y) - \rho(u, v)| \leq \rho(x, u) + \rho(v, y)$.

8.1.3. **(a)** We must show that σ is a metric on A. It is obvious that σ satisfies Definition 8.1.1**(a)** and **(b)**. For **(c)**,

$$\begin{aligned}
\sigma(u, v) &= \frac{\rho(u, v)}{1 + \rho(u, v)} = \frac{1}{1 + \dfrac{1}{\rho(u, v)}} \leq \frac{1}{1 + \dfrac{1}{\rho(u, w) + \rho(w, v)}} \\
&= \frac{\rho(u, w) + \rho(w, v)}{1 + \rho(u, w) + \rho(w, v)} = \frac{\rho(u, w)}{1 + \rho(u, w) + \rho(w, v)} + \frac{\rho(w, v)}{1 + \rho(u, w) + \rho(w, v)} \\
&\leq \frac{\rho(u, w)}{1 + \rho(u, w)} + \frac{\rho(w, v)}{1 + \rho(w, v)} = \sigma(u, w) + \sigma(w, v).
\end{aligned}$$

(b) Define $\rho_{n+1}(u, v) = \dfrac{\rho_n(u, v)}{1 + \rho_n(u, v)}$, $n \geq 1$. Then (A, ρ_n) is a metric space, by induction.

8.1.4. Suppose that $u_0 \in S$. If S is open in (A, ρ) there is an $\epsilon > 0$ such that $\left\{u \,\middle|\, \rho(u, u_0) < \epsilon\right\} \subset S$. If $\sigma(u, u_0) < \dfrac{\epsilon}{1 + \epsilon}$, then $\dfrac{\rho(u, u_0)}{1 + \rho(u, u_0)} < \dfrac{\epsilon}{1 + \epsilon}$, so $\rho(u, u_0) < \epsilon$ and therefore $u \in S$; hence, S is open in (A, σ).

If S is open in (A,σ) there is an $\epsilon > 0$ such that $\{u \mid \sigma(u,u_0) < \epsilon\} \subset S$. We may assume that $\epsilon < 1$. If $\rho(u,u_0) < \dfrac{\epsilon}{1-\epsilon}$, then $\sigma(u,u_0) < \epsilon$ and therefore $u \in S$; hence, S is open in (A,ρ).

8.1.5. It is obvious that ρ satisfies Definition 8.1.1**(a)** and **(b)**. For **(c)**, we consider the possible cases:

(i) If $u \ne w$, then $\rho(u,v) \le 1 = \rho(u,w) \le \rho(u,w) + \rho(w,v)$, since $\rho(w,v) \ge 0$.

(ii) If $u = w$ and $v \ne u$, then $\rho(u,v) \le 1 = 0 + 1 = \rho(u,w) + \rho(w,v)$.

(iii) If $u = v = w$, then $\rho(u,v) = 0 = 0 + 0 = \rho(u,w) + \rho(w,v)$.

8.1.6. **(a)** If $\rho(v,u_0) = r_1 > r$ and $\rho(u,u_0) < r$, then $\rho(v,u_0) \le \rho(v,u) + \rho(u,u_0)$, so $\rho(v,u) \ge \rho(v,u_0) - \rho(u,u_0) > r_1 - r$; hence, $S_\epsilon(v) \cap S_r(u_0) = \emptyset$ if $\epsilon \le r_1 - r$.

(b) $\overline{S}_1(u_0) = \{u_0\}$, $\{u \mid \rho(u,u_0) \le 1\} = A$

8.1.7. **(a)** If $S_1, S_2, \dots, S_n$ are open and $u \in \cap_{i=1}^n S_i$, there are positive numbers $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ such that $S_{\epsilon_i}(u) \subset S_i$. If $\epsilon = \min\{\epsilon_i \mid 1 \le i \le n\}$, then $S_\epsilon(u) \subset \cap_{i=1}^n S_i$.

(b) Let $T = \cup_{i=1}^n T_i$ where $T_1, T_2, \dots, T_n$ are closed. Then $T^c = \cap_{i=1}^n T_i^c$. Since T_i^c is open, so is T^c, by **(a)**. Hence, T is closed.

8.1.8. **(a)** Since U is a neighborhood of u_0 there is an $\epsilon > 0$ such that $S_\epsilon(u_0) \subset U$. Since $U \subset V$, $S_\epsilon(u_0) \subset V$. Hence, V is a neighborhood of u_0.

(b) Since $U_1, U_2, \dots, U_n$ are neighborhoods of u_0 there are positive numbers $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ such that $S_{\epsilon_i}(u_0) \subset U_i$, $1 \le i \le n$. If $\epsilon = \min\{\epsilon_i \mid 1 \le i \le n\}$, then $S_\epsilon(u_0) \subset \cap_{i=1}^n U_i$. Hence, $\cap_{i=1}^n U_i$ is a neighborhood of u_0.

8.1.9. If u_0 is a limit point of S then every neighborhood of u_0 contains points of S other than u_0. If every neighborhood of u_0 also contains a point in S^c, then $u_0 \in \partial S$. If there is a neighborhood of u_0 that does not contain a point in S^c then $u_0 \in S^0$. These are the only possibilities.

8.1.10. An isolated point u_0 of S has a neighborhood V that contains no other points of S. Any neighborhood U of u_0 contains $V \cap U$, also a neighborhood of u_0 (Exercise 8.1.8**(b)**), so $S^c \cap U \ne \emptyset$. Since $u_0 \in S \cap U$, $u_0 \in \partial S$.

8.1.11. **(a)** If $u_0 \in \partial S$ and U is a neighborhood of u_0, then (A) $U \cap S \ne \emptyset$. If u_0 is not a limit point of S then (B) $U \cap (S - \{u_0\}) = \emptyset$ for some U. Now (A) and (B) imply that $u_0 \in S$, and (B) implies that u_0 is an isolated point of S.

(b) If S is closed, Theorem 8.1.13 and **(a)** imply that $\partial S \subset S$; hence, $\overline{S} = S \cup \partial S = S$. If $\overline{S} = S$, then $\partial S \subset S$. Since $S^0 \subset S$, S is closed, by Exercise 8.1.9 and Theorem 8.1.13.

8.1.12. **(a)** If u_0 is a limit point of ∂S and $\epsilon > 0$, there is a u_1 in $S_\epsilon(u_0) \cap \partial S$. Since $S_\epsilon(u_0)$ is a neighborhood of u_1 and $u_1 \in \partial S$, $S_\epsilon(u_0) \cap S \ne \emptyset$ and $S_\epsilon(u_0) \cap S^c \ne \emptyset$. Therefore, u_0 is in ∂S and ∂S is closed (Theorem 8.1.13).

(b) If $u_0 \in S^0$, then $S_\epsilon(u_0) \subset S$ for some $\epsilon > 0$. Since $S_\epsilon(u_0) \subset S^0$ (Example 8.1.5) S^0 is open.

(c) Apply **(b)** to S^c.

(d) If u_0 is not a limit point of S, there is a neighborhood of u_0 that contains no points of S distinct from u_0. Therefore, the set of points that are not limit points of S is open and the set of limit points of S is consequently closed.

(e) $\left(\overline{S}\right)^c =$ exterior of S, which is open, by **(b)**. Hence, $\overline{S}$ is closed and $\overline{\left(\overline{S}\right)} = \overline{S}$ from (Exercise 8.1.11**(b)**), applied to $\overline{S}$.

(a) $u \in (S_1 \cap S_2)^0 \Leftrightarrow u$ has a neighborhood $N \subset S_1 \cap S_2 \Leftrightarrow u \in S_1^0$ and $u \in S_2^0 \Leftrightarrow u \in S_1^0 \cap S_2^0$. **(b)** $u \in S_1^0 \cup S_2^0 \Rightarrow u \in S_1^0$ or $u \in S_2^0 \Rightarrow u$ has a neighborhood N such that $N \subset S_1$ or $N \subset S_2 \Rightarrow u$ has a neighborhood $N \subset S_1 \cup S_2 \Rightarrow u \in (S_1 \cup S_2)^0$.

8.1.14. **(a)** $u \in \partial(S_1 \cup S_2) \Rightarrow$ every neighborhood of u contains a point in $(S_1 \cup S_2)^c$ and a point in $S_1 \cup S_2$. If every neighborhood of u contains points in $S_1 \cap S_2$, then $u \in \partial S_1 \cap \partial S_2 \subset \partial S_1 \cup \partial S_2$. Now suppose that u has a neighborhood N such that $N \cap S_1 = \emptyset$. If U is any neighborhood of u, then so is $N \cap U$, and $N \cap U \cap S_2 \neq \emptyset$, since $N \cap U$ must intersect $S_1 \cup S_2$. This means that $u \in \partial S_2 \subset \partial S_1 \cup \partial S_2$. A similar argument applies if u has a neighborhood N such that $N \cap S_2 = \emptyset$.

(b) $u \in \partial(S_1 \cap S_2) \Rightarrow$ every neighborhood of u contains a point in $(S_1 \cap S_2)^c$ and a point in $S_1 \cap S_2$. If every neighborhood of u contains a point in $(S_1 \cup S_2)^c$, then $u \in \partial S_1 \cap \partial S_2 \subset \partial S_1 \cup \partial S_2$. Now suppose that u has a neighborhood N such that $N \subset S_1$. If U is any neighborhood of u, then so is $N \cap U$, and $N \cap U \cap S_2^c \neq \emptyset$, since $N \cap U$ must intersect $(S_1 \cap S_2)^c$. This means that $u \in \partial S_2 \subset \partial S_1 \cup \partial S_2$. A similar argument applies if u has a neighborhood N such that $N \subset S_2$.

(c) If $u \in \partial\overline{S}$, then any neighborhood N of u contains points u_0 in $\overline{S}$ and u_1 not in $\overline{S}$. Either $u_0 \in S$ or $u_0 \in \partial S$. In either case $N \cap S \neq \emptyset$. Since $u_1 \in N \cap S^c$, it follows that $u \in \partial S$; hence, $\partial\overline{S} \subset \partial S$.

(d) Obvious from the definition of ∂S.

(e)

$$\begin{aligned} \partial(S - T) &= \partial(S \cap T^c) && \text{(definition of } S - T) \\ &\subset \partial S \cup \partial T^c && \text{(Exercise 8.1.14(b))} \\ &= \partial S \cup \partial T && \text{(Exercise 8.1.14(d)).} \end{aligned}$$

8.1.15. It is obvious that $\|\cdot\|_n$ satisfies Definition 8.1.3**(a)** and **(b)**. For **(c)**, since $|x_i + y_i| \le |x_i| + |y_i|$, $i = 1, 2, \dots, n$, it follows that

$$\max\left\{|x_i + y_i| \,\middle|\, 1 \le i \le n\right\} \le \max\left\{|x_i| \,\middle|\, 1 \le i \le n\right\} + \max\left\{|y_i| \,\middle|\, 1 \le i \le n\right\};$$

hence, $\|\mathbf{X} + \mathbf{Y}\|_n \le \|\mathbf{X}\|_n + \|\mathbf{Y}\|_n$.

8.1.16. **(a)** $\rho(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{k} \rho(x_i, y_i) \ge 0$ with equality if and only if $x_i = y_i$ for $1 \le i \le k$; that is $\mathbf{X} = \mathbf{Y}$;

$$\rho(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{k} \rho_i(x_i, y_i) = \sum_{i=1}^{k} \rho_i(y_i, x_i) = \rho(\mathbf{Y}, \mathbf{X});$$

$$\rho(\mathbf{X}, \mathbf{Z}) = \sum_{i=1}^{k} \rho(x_i, z_i) \le \sum_{i=1}^{k} (\rho(x_i, y_i) + \rho(y_i, z_i)) = \rho(\mathbf{X}, \mathbf{Y}) + \rho(\mathbf{Y}, \mathbf{Z}).$$

(b) $\lim_{r\to\infty}\mathbf{X}_r = \widehat{\mathbf{X}} \Leftrightarrow \lim_{r\to\infty}\rho(\mathbf{X}_r,\widehat{\mathbf{X}}) = 0 \Leftrightarrow \lim_{r\to\infty}\rho_i(x_{ir},\widehat{x}_i) = 0,\ 1 \le i \le k$ $\Leftrightarrow \lim_{r\to\infty} x_{ir} = \widehat{x}_i,\ 1 \le i \le k$.

(c) Let $\epsilon > 0$. If $\{\mathbf{X}_r\}_{r=1}^\infty$ is a Cauchy sequence there is an integer m such that $\rho(\mathbf{X}_r,\mathbf{X}_s) < \epsilon$ if $r, s \ge m$. Therefore, for $1 \le i \le k$, $\rho(x_{ir},x_{is}) < \epsilon$ for $r, s \ge m$, so $\{x_{ir}\}_{r=1}^\infty$ is a Cauchy sequence.

Conversely, if $\{x_{ir}\}_{r=1}^\infty$ is a Cauchy sequence for $\le i \le k$, there is an integer m such that $\rho(x_{ir},x_{is}) < \epsilon/k$, $1 \le i \le k$. Therefore, $\rho(\mathbf{X}_r,\mathbf{X}_s) < \epsilon$ if $r, s \ge k$, so $\{\mathbf{X}_r\}_{r=1}^\infty$ is a Cauchy sequence.

(d) Follows from **(b)** and **(c)**.

8.1.17. **(a)** Since $\sum_{i=1}^\infty \alpha_i < \infty$, $\rho(\mathbf{X},\mathbf{Y})$ is defined for all $(\mathbf{X},\mathbf{Y})$, by the comparison test. For the verification that ρ is a metric, see the solution of Exercise 8.1.3(**a**).

(b) . Suppose that $\lim_{r\to\infty} x_{ir} = \widehat{x}_i$, $i \ge 1$. Let $\epsilon > 0$. Choose N so that $\sum_{i=N+1}^\infty \alpha_i < \epsilon/2$. Now choose R so that $\rho(x_{ir},\widehat{x}_i) < \dfrac{\epsilon}{2N\alpha_i}$ if $r \ge R$, $1 \le i \le N$. Then $\rho(\mathbf{X}_r,\widehat{\mathbf{X}}) < \epsilon$ if $r \ge R$, so $\lim_{r\to\infty}\mathbf{X}_r = \widehat{\mathbf{X}}$.

Now suppose that $\lim_{r\to\infty}\mathbf{X}_r = \widehat{\mathbf{X}}$. Let $0 < \epsilon < 1$. For a fixed $i \ge 1$, choose R_i so that $\rho(\mathbf{X}_r,\mathbf{X}) < \dfrac{\epsilon\alpha_i}{2}$, $r \ge R_i$. Then $\dfrac{\rho(x_{ir},\widehat{x}_i)}{1+\rho(x_{ir},\widehat{x}_i)} < \dfrac{\epsilon}{2}$, $r \ge R_i$. This implies that $\rho(x_{ir},\widehat{x}_i) < \dfrac{\epsilon}{2-\epsilon} < \epsilon$, $r \ge R_i$, so $\lim_{r\to\infty} x_{ir} = \widehat{x}_i$.

(c) Suppose that $\{x_{ir}\}_{r=1}^\infty$ is a Cauchy sequence for all $i \ge 1$. Let $\epsilon > 0$. Choose N so that $\sum_{i=N+1}^\infty \alpha_i < \epsilon/2$. Now choose R so that $\rho(x_{ir},\widehat{x}_{is}) < \dfrac{\epsilon}{2N\alpha_i}$ if $r, s \ge R$, $1 \le i \le N$. Then $\rho(\mathbf{X}_r,\widehat{\mathbf{X}}_s) < \epsilon$ if $r, s \ge R$, so $\{\mathbf{X}_r\}_{r=1}^\infty$ is a Cauchy sequence.

Now suppose that $\{\mathbf{X}_r\}$ is a Cauchy sequence. Let $0 < \epsilon < 1$. For a fixed $i \ge 1$, choose R_i so that $\rho(\mathbf{X}_r,\mathbf{X}_s) < \dfrac{\epsilon\alpha_i}{2}$, $r, s \ge R_i$. Then $\dfrac{\rho(x_{ir},x_{is})}{1+\rho(x_{ir},x_{is})} < \dfrac{\epsilon}{2}$, $r, s \ge R_i$. This implies that $\rho(x_{ir},x_{is}) < \dfrac{\epsilon}{2-\epsilon} < \epsilon$, $r, s \ge R_i$, so $\{x_{ir}\}$ is a Cauchy sequence.

(d) Follows from **(b)** and **(c)**.

8.1.20. Let a and b be nonnegative. Since $p < 1$, Minkowski's inequality implies that $(a^{1/p}+b^{1/p})^p \le a+b$. By letting $u = a^{1/p}$ and $v = b^{1/p}$ we see that $(u+v)^p \le u^p+v^p$ if $u, v > 0$. Therefore,

$$\sum_{i=1}^n |x_i + y_i|^p \le \sum_{i=1}^n (|x_i| + |y_i|)^p \le \sum_{i=1}^n |x_i|^p + \sum_{i=1}^p |y_i|^p,$$

so $\rho(\mathbf{X}+\mathbf{Y}) \le \rho\mathbf{X} + \rho\mathbf{Y}$. However, ρ is not a norm, since $\rho(c\mathbf{X}) = c^p\rho(\mathbf{X})$.

8.1.21. **(a)** If $\sum |x_i|^p < \infty$ then $\lim_{i\to\infty} x_i = 0$, so there is an integer J such that $|x_i| < 1$ for $i > J$. If $r > p$, then $|x_i|^r < |x_i|^p$ if $i > J$; hence, $\sum |x_i|^r < \infty$, by the comparison test.

(b) For $r \ge p$ let $\sigma(r) = \left(\sum_{i=1}^{\infty} |x_i|^r\right)^{1/r}$. Since $|x_i|/\sigma(r) \le 1$ if $r \ge p$, $(|x_i|/\sigma(r))^p \ge (|x_i|/\sigma(r))^r$; therefore,

$$\frac{\sigma(p)}{\sigma(r)} = \left(\sum_{i=1}^{\infty}\left(\frac{|x_i|}{\sigma(r)}\right)^p\right)^{1/p} \ge \left(\sum_{i=1}^{\infty}\left(\frac{|x_i|}{\sigma(r)}\right)^r\right)^{1/p} = 1.$$

(c) Let $r \ge p > 1$. Since $1/r < 1$, Exercise 8.1.20 with p replaced by $1/r$, $u = \sum_{i=1}^{J} |x_i|^r$, and $v = \sum_{i=J+1}^{\infty} |x_i|^r$ implies that

$$\|\mathbf{X}\|_r \le \left(\sum_{i=1}^{J} |x_i|^r\right)^{1/r} + \left(\sum_{i=J+1}^{\infty} |x_i|^r\right)^{1/r}. \qquad \text{(A)}$$

Since $\lim_{i\to\infty} x_i = 0$, $|x_i| = \|\mathbf{X}\|_\infty$ for some i, so (B) $\|\mathbf{X}\|_\infty \le \|\mathbf{X}\|_r$. Let $\epsilon > 0$, and choose J so that $\left(\sum_{i=J+1}^{\infty} |x_i|^p\right)^{1/p} < \epsilon$. Applying **(b)** to $\widehat{\mathbf{X}} = \{0, \dots, 0, , x_{J+1}, x_{J+2}, \dots\}$ shows that $\left(\sum_{i=J+1}^{\infty} |x_i|^r\right)^{1/r} < \epsilon$ if $r \ge p$; therefore, from (A) and (B),

$$\|\mathbf{X}\|_\infty \le \|\mathbf{X}\|_r \le \left(\sum_{i=1}^{J} |x_i|^r\right)^{1/r} + \epsilon, \quad r \ge p,$$

so

$$\|\mathbf{X}\|_\infty \le \|\mathbf{X}\|_r \le J^{1/r}\|\mathbf{X}\|_\infty + \epsilon.$$

Letting $r \to \infty$ yields

$$\|\mathbf{X}\| \le \varliminf_{r\to\infty} \|\mathbf{X}\|_r\| \le \varlimsup_{r\to\infty} \|\mathbf{X}\|_r \le \|\mathbf{X}\|_\infty + \epsilon.$$

Since ϵ is arbitrary, $\lim_{r\to\infty} \|\mathbf{X}\|_r = \|\mathbf{X}\|_\infty$.

8.1.22. **(a)** From Exercise 8.1.3 $|\rho(u,v) - \rho(u_n,v_n)| \le \rho(u,u_n) + \rho(v,v_n)$. If $\epsilon > 0$ there is an integer k such that $\rho(u,u_n) + \rho(v,v_n) < \epsilon$ if $n \ge k$; hence, $|\rho(u,v) - \rho(u_n,v_n)| < \epsilon$ if $n \ge k$, so $\lim_{n\to\infty} \rho(u_n,v_n) = \rho(u,v)$.

(b) Let $v_n = v$ and apply **(a)**

8.1.23. There is an integer k such that $\|u_r - u_s\| < 1$ if $r, s \geq k$. Therefore, if $r \geq k$, then $\|u_r\| \leq \|u_k\| + \|u_r - u_k\| \leq \|u_k\| + 1$.

8.1.24. **(a)** Clearly $\|\mathbf{X}\| \geq 0$. If $\|\mathbf{X}\| = 0$, then $x_1 = 0$. Now suppose that $n > 1$ and $x_i = 0$, $1 \leq i \leq n-1$. Since $\sum_{i=1}^{n} x_i = 0$, it follows that $x_n = 0$. Therefore, $\mathbf{X} = \mathbf{0}$, by induction. Obviously $\|a\mathbf{X}\| = |a|\|\mathbf{X}\|$.

By the triangle inequality for real numbers,

$$\left|\sum_{i=1}^{n}(x_i + y_i)\right| = \left|\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right| \leq \left|\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right| \leq \left|\sum_{i=1}^{n} x_i\right| + \left|\sum_{i=1}^{n} y_i\right|,$$

so $\mathbf{X}+\mathbf{Y} \in A$ if $\mathbf{X}, \mathbf{Y} \in A$ and $\|\mathbf{X}+\mathbf{Y}\| \leq \|\mathbf{X}\| + \|\mathbf{Y}\|$. The verification of the other vector space properties is straightforward.

(b) Let $\{\mathbf{X}_r\}_{r=1}^{\infty}$ with $\mathbf{X}_r = \{x_{ir}\}_{i=1}^{\infty}$ be a Cauchy sequence in A. If $\epsilon > 0$, there is an integer k such that $\|\mathbf{X}_r - \mathbf{X}_s\| < \epsilon/2$ if $r, s \geq k$. Since $x_{nr} - x_{ns} = \sum_{i=1}^{n}(x_{ir} - x_{is}) - \sum_{i=1}^{n-1}(x_{ir} - x_{is})$, $|x_{nr} - x_{ns}| < \epsilon$ for all $n \geq 1$ if $r, s \geq k$. Hence, $\widehat{x}_n = \lim_{r\to\infty} x_{nr}$ exists for all $n \geq 1$ (Theorem 4.1.13). Let $\widehat{\mathbf{X}} = \{\widehat{x}_i\}_{i=1}^{\infty}$.

By Exercise 8.1.23, there is a constant M such that $\|\mathbf{X}_r\| \leq M$ for all $r \geq 1$; that is, $\left|\sum_{i=1}^{n} x_{ir}\right| \leq M$ if $r, n \geq 1$. Letting $r \to \infty$ for each fixed n shows that $\widehat{\mathbf{X}} \in A$, and $\|\widehat{\mathbf{X}}\| \leq M$.

Since $\left|\sum_{i=1}^{n}(x_{ir} - x_{is})\right| < \epsilon/2$ for all $n \geq 1$ if $r, s \geq k$, letting $s \to \infty$ shows that $\left|\sum_{i=1}^{n}(x_{ir} - \widehat{x}_i)\right| \leq \epsilon/2$ for all $n \geq 1$ if $r \geq k$; hence, $\lim_{r\to\infty} \|\mathbf{X}_r - \widehat{\mathbf{X}}\| = 0$.

8.1.25. **(a)** Straightforward.

(b) If $m > n$ then $|f_n(x) - f_m(x)| = f_n(x)(1 - f_{m-n}(x)) \leq f_n(x)$, so $\|f_n - f_m\| \leq \int_a^b f_n(x)\,dx \leq (b-a)/(n+1)$.

(c) $\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & a \leq x < b \\ 1, & x = b, \end{cases}$ is discontinuous, and therefore not in $C[a,b]$

8.1.26. **(a)** Since $|x_i + y_i| \leq |x_i| + |y_i|$, $\sup\left\{|x_i + y_i| \,\middle|\, i \geq 1\right\} \leq \sup\left\{|x_i| \,\middle|\, i \geq 1\right\} + \sup\left\{|y_i| \,\middle|\, i \geq 1\right\}$.

(b) Let $\{\mathbf{X}_r\}_{r=1}^{\infty}$ be a Cauchy sequence in ℓ_∞. Write $\mathbf{X}_r = \{x_{ir}\}_{i=1}^{\infty}$. If $\epsilon > 0$ there is an integer R (independent of i) such that $|x_{ir} - x_{is}| < \epsilon$ if $r, s \geq R$; that is, $\{x_{ir}\}_{r=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed $i \geq 1$. Let $\lim_{r\to\infty} x_{ir} = \widehat{x}_i$, $i \geq 1$, and let $\widehat{X} = \{\widehat{x}_i\}_{i=1}^{\infty}$. Then $\lim_{r\to\infty} X_r = \widehat{X}$. To see this, let ϵ and R be as above. For each i, choose $s_i > R$ so that $|x_{is_i} - \widehat{x}_i| < \epsilon$. Then $|x_{ir} - \widehat{x}_i| \le |x_{ir} - x_{is_i}| + |x_{is_i} - \widehat{x}_i| < 2\epsilon$ if $r > R$; that is (A) $\|\mathbf{X}_r - \widehat{\mathbf{X}}\| < 2\epsilon$ if $r \ge R$. Finally, since $\|\widehat{\mathbf{X}}\| \le \|\widehat{\mathbf{X}} - \mathbf{X}_r\| + \|\mathbf{X}_r\|$ and $\{\mathbf{X}_r\}_{r=1}^{\infty}$ is bounded (Exercise 8.1.23), (A) implies that $\mathbf{X} \in \ell_\infty$.

8.1.27. Since the set of convergent sequences is closed under addition and scalar multiplication (Theorem 4.1.8), A is a subspace of $\mathbb{R}^\infty$. Since $\|\cdot\| = \|\cdot\|_\infty$, $(A, \|\cdot\|)$ is a normed subspace of ℓ_∞, which is complete (Exercise 8.1.26**(b)**). Therefore, if $\{\mathbf{X}_r\}_{r=1}^{\infty}$ is a Cauchy sequence in $(A, \|\cdot\|)$, there is an $\widehat{\mathbf{X}}$ such that $\lim_{r\to\infty} \mathbf{X}_r = \widehat{\mathbf{X}} = \{\widehat{x}_i\}_{i=1}^{\infty}$ in the sense that (A) $\lim_{r\to\infty} \|\mathbf{X}_r - \widehat{\mathbf{X}}\|_\infty = 0$. We have only to show that $\widehat{\mathbf{X}} = \{\widehat{x}_i\}_{i=1}^{\infty} \in A$; that is, that $\lim_{i\to\infty} \widehat{x}_i$ exists. By the triangle inequality $|\widehat{x}_i - \widehat{x}_j| \le |\widehat{x}_i - x_{ir}| + |x_{ir} - x_{jr}| + |x_{jr} - \widehat{x}_j|$. Suppose that $\epsilon > 0$. From (A), there is an integer k such that $|\widehat{x}_i - x_{ik}| < \epsilon/3$ and $|x_{jk} - \widehat{x}_j| < \epsilon/3$ for all $i, j \ge 1$. Therefore, $|\widehat{x}_i - \widehat{x}_j| < |x_{ik} - x_{jk}| + 2\epsilon/3$ for all $i, j \ge 1$. Since $\mathbf{X}_k$ converges, it is a Cauchy sequence (Theorem 4.1.13), so there is an integer N such that $|x_{ik} - x_{jk}| < \epsilon/3$ if $i, j \ge N$. Therefore, $|\widehat{x}_i - \widehat{x}_j| < \epsilon$ if $i, j \ge k$. Hence, $\{\widehat{x}_i\}_{i=1}^{\infty}$ is a Cauchy sequence, and therefore convergent (Theorem 4.1.13).

8.1.28. Since the set of sequences that converge to zero is closed under addition and scalar multiplication (Theorem 4.1.8), A is a subspace of $\mathbb{R}^\infty$. If $\mathbf{X} = \{x_i\}_{i=1}^{\infty}$ is in A and $\|\mathbf{X}\|_\infty = 0$, then $x_i = 0$ for all i. If $\|\mathbf{X}\|_\infty > 0$, there is an integer k such that $|x_i| < \|\mathbf{X}\|_\infty$ if $i > k$, so $\|X\|_\infty = \max\left\{|x_i| \,\middle|\, 1 \le i \le k\right\}$. In either case $\|\mathbf{X}\| = \|\mathbf{X}\|_\infty$. Now an argument similar to the one used in the solution of Exercise 8.1.27 shows that if $\{\mathbf{X}_r\}_{r=1}^{\infty}$ is a Cauchy sequence in A, there is a sequence $\widehat{\mathbf{X}} = \{\widehat{x}_i\}_{i=1}^{\infty}$ in ℓ_∞ such that (A) $\lim_{r\to\infty} \|\mathbf{X}_r - \widehat{\mathbf{X}}\|_\infty = 0$. Let $\overline{x} = \lim_{i\to\infty} \widehat{x}_i$. Then (B) $|\overline{x}| \le |\overline{x} - \widehat{x}_i| + |\widehat{x}_i - x_{ir}| + |x_{ir}|$. Let $\epsilon > 0$. From (A), there is an integer r such that $|\widehat{x}_i - x_{ir}| < \epsilon$ for all $i \ge 1$. Hence, (C) $|\overline{x}| \le |\overline{x} - \widehat{x}_i| + |x_{ir}| + \epsilon$. Since $\lim_{i\to\infty} x_{ir} = 0$ and $\lim_{i\to\infty} \widehat{x}_i = \overline{x}$, we can choose i so large that $|x_{ir}| < \epsilon$ and $|\overline{x} - \widehat{x}_i| < \epsilon$. Now (C) implies that $|\overline{x}| < 3\epsilon$. Since ϵ is arbitrary, $\overline{x} = 0$.

8.1.29. **(a)** Similar to (and simpler than) the following argument for **(b)**.

(b) Let $\{\mathbf{X}_r\}_{r=1}^{\infty}$ be a Cauchy sequence in ℓ_p, with $\mathbf{X}_r = \{x_{ir}\}_{i=1}^{\infty}$. If $\epsilon > 0$, there is an integer k such that (A) $\|\mathbf{X}_r - \mathbf{X}_s\|_p < \epsilon$ if $r, s \ge k$. Therefore, $|x_{ir} - x_{is}| < \epsilon$ for all $i \ge 1$ if $r, s \ge k$. Hence, $\widehat{x}_i = \lim_{r\to\infty} x_{ir}$ exists for $i \ge 1$ (Theorem 4.1.13). Let $\widehat{\mathbf{X}} = \{\widehat{x}_i\}_{i=1}^{\infty}$. By Exercise 8.1.23, there is a constant M such that $\|\mathbf{X}_r\|_p \le M$ for all $r \ge 1$; that is, $\left(\sum_{i=1}^{\infty} |x_{ir}|^p\right)^{1/p} \le M$ for $r \ge 1$. Therefore, for every integer n, $\sum_{i=1}^{n} |x_{ir}|^p \le M^p$ for all $r \ge 1$. Letting $r \to \infty$ with n fixed yields $\sum_{i=1}^{n} |\widehat{x}_i|^p \le M^p$. Now letting $n \to \infty$ shows that $\sum_{i=1}^{\infty} |\widehat{x}_i|^p \le M^p$, so $\widehat{X} \in \ell_p$.

From (A), if n is a positive integer, then $\sum_{i=1}^{n} |x_{ir} - x_{is}|^p < \epsilon^p$ if $r, s \ge k$. Letting $s \to \infty$

here yields $\sum_{i=1}^{n} |x_{ir} - \widehat{x}_i|^p \le \epsilon^p$ if $r \ge$. Now letting $n \to \infty$ shows that $\|\mathbf{X}_r - \widehat{\mathbf{X}}\| < \epsilon$ if $r \ge$. Hence, $\lim_{r\to\infty} \mathbf{X}_r = \widehat{\mathbf{X}}$.

8.2 COMPACT SETS IN A METRIC SPACE

8.2.1. If H is an open covering of $\cup_{j=1}^{k} T_j$ then H is an open covering of T_j, $1 \le j \le k$. Since T_j is compact, T_j has a finite subcovering $H_j \subset H$, and $\cup_{j=1}^{k} H_j$ is a finite open covering of $\cup_{j=1}^{k} T_j$.

8.2.2. **(a)** Let S be a closed subset of a compact set T. Let $\{s_n\}$ be a sequence in S. Since T is compact there is a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $\lim_{j\to\infty} s_{n_j} = \overline{t} \in T$. Either $\overline{t} = s_{n_j}$ for some j or $\overline{t}$ is a limit point of $\{s_n\}$ and therefore in S, since S is closed. Hence, S is compact (Theorem 8.2.4).

(b) $\cap \left\{T \mid T \in T\right\}$ is a closed subset of $\widehat{T}$. Apply **(a)**.

(c) Since every T in T is closed (Theorem 8.2.6), $\cap \left\{T \mid T \in T\right\}$ is closed. Apply **(b)**.

8.2.3. If $a \in S \cap T$ then $\text{dist}(S,T) = \rho(a,a) = 0$. Now assume that $S \cap T = \emptyset$. For each positive integer n there is an s_n in S and a t_n in T such that $\rho(s_n,t_n) < \text{dist}(S,T) + 1/n$. If $\rho(s_n,t_n) = \text{dist}(S,T)$ for some n, we are finished. If $\rho(s_n,t_n) > \text{dist}(S,T)$ for all n, then at least one of the sequences $\{s_n\}$, $\{t_n\}$ has infinitely many distinct terms. Suppose that $\{s_n\}$ has this property. Since S is compact, there is a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\lim_{k\to\infty} s_{n_k} = \overline{s} \in S$ (Theorem 8.2.3). From (A), $\lim_{k\to\infty} \rho(s_{n_k}, t_{n_k}) = \text{dist}(S,T)$. Let $s_k' = s_{n_k}$ and $t_k' = t_{n_k}$. Then $\lim_{k\to\infty} s_k' = \overline{s}$ and $\lim_{k\to\infty} \rho(s_k', t_k') = \text{dist}(S,T)$. Since T is compact, there is a subsequence $\{t_{k_j}'\}$ of $\{t_k'\}$ such that $\lim_{j\to\infty} t_{n_j}' = \overline{t} \in T$ (Theorem 8.2.3). Since any subsequence of a convergent sequence converges to the limit of the sequence, $\lim_{j\to\infty} s_{n_j}' = \overline{s}$ and $\lim_{j\to\infty} \rho(s_{k_j}', t_{k_j}') = \text{dist}(S,T)$. Since $\lim_{j\to\infty} \rho(s_{k_j}', t_{k_j}') = \rho(\overline{s}, \overline{t})$ (Exercise ???**(a)**), $\rho(\overline{s}, \overline{t}) = \text{dist}(S,T)$.

8.2.4. **(a)** If T is totally bounded, there is a finite set $T_1 = \{\mathbf{X}_1, \dots, \mathbf{X}_k\}$ of bounded sequences such that if $\mathbf{X} \in T$ then $\rho(\mathbf{X}, \mathbf{X}_i) < 1$ for some $i \in \{1, \dots, k\}$. If $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary members of T, then $\rho(\mathbf{X}, \mathbf{X}_r) < 1$ and $\rho(\mathbf{X}, \mathbf{X}_s) < 1$ for some r and s in $\{1, 2, \dots, n\}$, so

$$\begin{aligned}\rho(\mathbf{X},\mathbf{Y}) &\le \rho(\mathbf{X},\mathbf{X}_r) + \rho(\mathbf{X}_r,\mathbf{X}_s) + \rho(\mathbf{X}_s,\mathbf{Y}) \\ &\le 2 + \rho(\mathbf{X}_r,\mathbf{X}_s) \le 2 + \max\left\{\rho(\mathbf{X}_i,\mathbf{X}_j) \mid 1 \le i < j \le n\right\}.\end{aligned}$$

Therefore, T is bounded.

(b) If $r \ne s$, then $\rho(\mathbf{X}_r, \mathbf{X}_s) = 1$, so T is bounded. Now suppose that $\epsilon < 1/2$ and $\mathbf{Y} = \{y_i\}_{i=1}^{\infty}$ is a sequence such that $\rho(\mathbf{X}_r, \mathbf{Y}) < \epsilon$. Then $y_r > 1 - \epsilon > 1/2$, so $\rho(\mathbf{X}_s, \mathbf{Y}) > 1/2$ for all $s \ne r$; that is, no vector can satisfy $\rho(\mathbf{X}_r, \mathbf{Y}) < \epsilon$ for more than one value of r. Hence, T is not totally bounded.

8.2.5. Since T is compact, T is bounded (Theorem 8.2.6). For each integer n there are members s_n and t_n of T such that (A) $d(T)-1/n < \rho(s_n,t_n) \le d(T)$. If $\rho(s_n,t_n) = d(T)$ for some n, we are finished. If $\rho(s_n,t_n) < d(T)$ for all n, then at least one of the sequences $\{s_n\}$, $\{t_n\}$ has infinitely many distinct terms. Suppose that $\{s_n\}$ has this property. Since T is compact, there is a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\lim_{k\to\infty} s_{n_k} = \overline{s} \in T$ (Theorem 8.2.3). From (A), $\lim_{k\to\infty} \rho(s_{n_k},t_{n_k}) = d(T)$. Let $s'_k = s_{n_k}$ and $t'_k = t_{n_k}$. Then $\lim_{k\to\infty} s'_k = \overline{s}$ and $\lim_{k\to\infty} \rho(s'_k,t'_k) = d(T)$. Since T is compact, there is a subsequence $\{t'_{k_j}\}$ of $\{t'_k\}$ such that $\lim_{j\to\infty} t'_{n_j} = \overline{t} \in T$ (Theorem 8.2.3). Since any subsequence of a convergent sequence converges to the limit of the sequence, $\lim_{j\to\infty} s'_{n_j} = \overline{s}$ and $\lim_{j\to\infty} \rho(s'_{k_j},t'_{k_j}) = d(T)$. Since $\lim_{j\to\infty} \rho(s'_{k_j},t'_{k_j}) = \rho(\overline{s},\overline{t})$ (Exercise ???**(a)**), $\rho(\overline{s},\overline{t}) = d(T)$.

8.2.6. Suppose that $\epsilon > 0$. Choose N so that $\sum_{i=N+1}^{\infty} \mu_i < \epsilon/2$. Let $\mu = \max\left\{\mu_i \,\middle|\, 1 \le i \le N\right\}$, and let p be an integer such that $p(\epsilon/2N) > \mu$. Let $Q_\epsilon = \left\{r_i\epsilon/2N \,\middle|\, r_i = \text{integer in}[-p,p]\right\}$. Then the set of sequences $\mathbf{X} = \{x_i\}_{i=1}^{\infty}$ in ℓ_1 such that $x_i \in Q_\epsilon$, $1 \le i \le N$, and $x_i = 0$, $i > N$, is a finite ϵ-net for T.

8.2.7. Suppose that $\epsilon > 0$. Choose J so that $\left(\sum_{i=J+1}^{\infty} |\mu_i|^2\right)^{1/2} < \epsilon$. Now let T^* be the subset of $\mathbb{R}^n$ such that $|x_i| \le \mu_i$, $1 \le i \le n$. Since T^* is a closed and bounded subset of $\mathbb{R}^n_2$, T^* is compact. Therefore, there is a finite collection of vectors $U^* = (u_1,u_2,\dots,u_n)$ that form a finite ϵ-net for T^*. The vectors $U = (u_1,u_2,\dots,u_n,0,\dots,0,\dots)$ form a 2ϵ-net for T.

8.2.8. Suppose that S is bounded. Let u_1 and be a fixed member of S and let u be an arbitrary member of S. Then $\rho(u,u_0) \le \rho(u,u_1) + \rho(u_1,u_0) \le d(S) + \rho(u_1,u_0)$, so D is bounded.

Now suppose that D is bounded; that is, there is a constant M such that $\rho(u,u_0) \le M$ if $u \in S$. Then, if u and v are both in S, $d(u,v) \le d(u,u_0) + d(v,u_0) \le 2M$, so S is bounded.

Let u_1 be an arbitrary member of S. Since $N_\epsilon(u_1)$ does not cover S there is a $u_2 \in S$ such that $\rho(u_1,u_2) \ge \epsilon$. Now suppose that $n \ge 2$ and we have chosen $u_1, u_2, \dots, u_n$ such that $\rho(u_i,u_j) \ge \epsilon$, $1 \le i < j \le n$. Since $\cup_{i=1}^n S\epsilon(u_i)$ does not cover S, there is $u_{n+1} \in S$ such that $\rho(u_i,u_{n+1}) \ge \epsilon$, $1 \le i \le n$. Therefore, $\rho(u_i,u_j) \ge \epsilon$, $1 \le i < j \le n+1$.

8.2.9. **(a)** If $u, v \in \overline{S}$ and $\epsilon > 0$, there are $s, t \in S$ such that $\rho(u,s) < \epsilon$ and $\rho(v,t) < \epsilon$. Therefore, $\rho(u,v) < \rho(u,s) + \rho(s,t) + \rho(t,v) < d(S) + 2\epsilon$. Since ϵ is arbitrary, $d(\overline{S}) \le d(S)$. Since $d(\overline{S}) \ge d(S)$ (obvious), $d(\overline{S}) = d(S)$.

(b) If $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in (A,ρ) then $S = \left\{x_n \,\middle|\, n \ge 1\right\}$ is bounded. From **(a)**, $\overline{S}$ is bounded. Therefore, $\overline{S}$ is compact (by assumption), so $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}$ such that $\lim_{j\to\infty} x_{n_j} = \overline{x} \in \overline{S}$. If $\epsilon > 0$, choose k so that $\rho(x_n,x_m) < \epsilon/2$, $m,n \ge k$. Now choose j so that $n_j \ge k$ and $\rho(x_m,\overline{x}) < \epsilon/2$. Then $\rho(x_n,\overline{x}) \le \rho(x_n,x_{n_j}) + \rho(x_{n_j},\overline{x}) < 2\epsilon$, $n \ge k$. Hence, $\lim_{n\to\infty} x_n = \overline{x}$.

8.2.10. Let $\{x_{ir}\}_{r=1}^{\infty}$ be a sequence in T_i, $1 \le i \le n$ and let $\mathbf{X}_r = (x_{1r}, x_{2r}, \dots, x_{kr})$. Suppose that T is compact. Then $\{\mathbf{X}_r\}$ has a subsequence $\{\mathbf{X}_{r_s}\}$ such that $\lim_{s\to\infty} \mathbf{X}_{r_s} = \widehat{\mathbf{X}} = (\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_k) \in T$ (Theorem 8.2.4). Therefore, $\lim_{s\to\infty} \rho(\mathbf{X}_{r_s}, \widehat{\mathbf{X}}) = 0$, so $\lim_{s\to\infty} \rho_i(x_{ir_s}, \widehat{x}_i) = 0$, $1 \le i \le k$. This implies that $\lim_{s\to\infty} x_{ir_s} = \widehat{x}_i \in T_i$, $1 \le i \le k$. Therefore, T_i is compact, $1 \le i \le k$ (Theorem 8.2.4).

Now suppose that $T_1, T_2, \dots, T_k$ are compact. We use induction on k. The result is trivial if $k = 1$. Now suppose that $k > 1$ and and the assertion is true with k replaced by $k-1$. Then is, $T_1 \times T_2 \times \cdots \times T_{k-1}$ is compact, so any sequence $\{\mathbf{X}_r\}_{r=1}^{\infty}$ in T has a sequence $\{\mathbf{X}_{r_s}\}_{s=1}^{\infty}$ such that $\lim_{s\to\infty} x_{ir_s} = \widehat{x}_i \in T_i$, $1 \le i \le k-1$. Denote $\mathbf{X}_s^{(1)} = \mathbf{X}_{r_s}$. Since T_k is compact, $\{x_{ks}^{(1)}\}_{s=1}^{\infty}$ has a subsequence $\{x_{ks_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} x_{ks_j} = \widehat{x}_k$ (Theorem 8.2.4). Since $\lim_{j\to\infty} x_{is_j} = \widehat{x}_i$, $1 \le i \le k-1$ (Theorem 4.2.2), $\lim_{j\to\infty} \mathbf{X}_{s_j} = (\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_k) \in T$. Hence, T is compact (Theorem 8.2.4).

8.2.11. For a fixed $j \ge 1$ let $\{x_{jr}\}_{r=1}^{\infty}$ be a sequence in T_j. For $i \ne j$ let $x_{ir} = \widehat{x}_i$, $r \ge 1$, where $\widehat{x}_i \in T_i$. Define $\mathbf{X}_r = \{x_{ir}\}_{i=1}^{\infty}$. Since T is compact, $\{\mathbf{X}_r\}_{r=1}^{\infty}$ has a subsequence $\{\mathbf{X}_{r_s}\}_{s=1}^{\infty}$ such that $\lim_{s\to\infty} \mathbf{X}_{r_s} = \widehat{\mathbf{X}} \in T$; that is, $\lim_{s\to\infty} \rho(\mathbf{X}_{r_s}, \widehat{\mathbf{X}}) = 0$. But $\rho(\mathbf{X}_{r_s}, \widehat{X}) = \alpha_j \dfrac{\rho_j(x_{jr_s}, \widehat{x}_j)}{1 + \rho_j(x_{jr_s}, \widehat{x}_j)}$, so $\lim_{s\to\infty} \rho_j(x_{jr_s}, \widehat{x}_j) = 0$; that is, $\lim_{s\to\infty} x_{jr_s} = \widehat{x}_j$. Hence, T_j is compact (Theorem 8.2.4).

8.2.12. If t_1 and t_2 are in $\cap_{n=1}^{\infty} T_n$ then t_1 and t_2 are in T_n for all $n \ge 1$, so $\rho(t_1, t_2) \le d(T_n)$ for $n \ge 1$; therefore, **(c)** implies that $\rho(t_1, t_2) = 0$; that is, $t_1 = t_2$. Hence, $\cap_{n=1}^{\infty} T_n$ contains at most one element. To complete the proof, we need only show that $\cap_{n=1}^{\infty} T_n \ne \emptyset$. We prove this by contradiction. If $\cap_{n=1}^{\infty} T_n = \emptyset$, then any t in T_1 is in the open set T_n^c for some n. Therefore, $T_1 \subset \cup_{n=1}^{\infty} T_n^c$. Since T_1 is compact, there is an integer N such that $T_1 \subset \cup_{n=1}^{N} T_n^c$. However, from **(b)**, $\cup_{n=1}^{N} T_n^c = T_N^c$, so $T_1 \subset T_N^c$. Since $T_{N+1} \subset T_1$, it follows that $T_{N+1} \subset T_N^c$. But $T_{N+1} \subset T_N$, from **(b)**. Since $T_{N+1} \ne \emptyset$, this is a contradiction.

8.3 CONTINUOUS FUNCTIONS ON METRIC SPACES

8.3.1. **(a)** $\Leftrightarrow$ **(b)**:

Assume **(a)** and let $u_0 \in f^{-1}(V)$; that is, $f(u_0) = v_0 \in V$. Since V is open, there is an $\epsilon > 0$ such that $N_\epsilon(v_0) \subset V$. Since f is continuous there is a $\delta > 0$ such that $f(N_\delta(u_0)) \subset N_\epsilon(v_0) \subset V$. This implies that $N_\delta(u_0) \subset f^{-1}(V)$, so $f^{-1}(V)$ is open.

Assume **(b)** and let $u_0 \in A$, $v_0 = f(u_0)$. If $\epsilon > 0$, then $u_0 \in f^{-1}(N_\epsilon(v_0))$, which is open, by assumption. Hence, there is a $\delta > 0$ such that $N_\delta(u_0) \subset f^{-1}(N_\epsilon(v_0))$, so $f(N_\delta(u_0)) \subset N_\epsilon(v_0)$, and f is continuous at u_0.

(b) $\Leftrightarrow$ **(c)**:

Assume **(b)** and let V be a closed subset of (B, ρ). Then V^c is open and $(f^{-1}(V))^c = f^{-1}(V^c)$ is open, so $f^{-1}(V))$ is closed.

Assume **(c)** and let V be an open subset of (B, ρ). Then V^c is closed and $(f^{-1}(V))^c =$

$f^{-1}(V^c)$ is closed, so $f^{-1}(V))$ is open.

8.3.2. Suppose that $B = B_1 \cup B_2$, where B_1 and B_2 are disjoint open sets. Then $A = A_1 \cup A_2$ where $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are open (Exercise 8.3.1) and disjoint. Since A is connected, $A_1 = \emptyset$ or $A_2 = \emptyset$.

8.3.3. **(a)** Since $f(S)$ is a compact subset of $\mathbb{R}$ (Theorem 8.3.6), it is bounded.

(b) For every n there is a $s_n \in S$ such that $\alpha \le s_n < \alpha + 1/n$. Since S is compact, $\{s_n\}$ has a convergent subsequence $\{s_{n_j}\}$ such that $\lim_{j\to\infty} s_{n_j} = u_1 \in S$. From Theorem 8.3.3, $f(u_1) = \lim_{j\to\infty} f(s_{n_j}) = \alpha$.

For every n there is a $t_n \in S$ such that $\beta - 1/n < t_n \le \beta$. Since S is compact, $\{t_n\}$ has a convergent subsequence $\{t_{n_j}\}$ such that $\lim_{j\to\infty} t_{n_j} = u_2 \in S$. From Theorem 8.3.3, $f(u_2) = \lim_{j\to\infty} f(t_{n_j}) = \beta$.

8.3.4. **(a)** Let $u, u_0 \in U$. By the triangle inequality, $\sigma(f(u), f(\overline{u})) \le \sigma(f(u), f(u_0)) + \sigma(f(u_0), f(\overline{u})$ and $\sigma(f(u_0), f(\overline{u})) \le \sigma(f(u_0), f(u)) + \sigma(f(u), f(\overline{u})$, so $g(u) - g(u_0) \le \sigma(f(u), f(u_0))$ and $g(u_0) - g(u) \le \sigma(f(u), f(u_0))$. Therefore, $|g(u) - g(u_0)| \le \sigma(f(u), f(u_0))$. Let $\epsilon > 0$. Since f is continuous on U there is a $\delta > 0$ such that $\sigma(f(u), f(u_0)) < \epsilon$ if $u \in N_\delta(u_0) \cap D_f$; hence, $|g(u) - g(u_0)| < \epsilon$ if $u \in N_\delta(u_0) \cap D_f$, so g is continuous on U.

(b) Theorem 8.3.8

(c) Exercise 8.3.3**(b)**.

8.3.5. Suppose that $u_0 \in D_f$ and let $\epsilon > 0$. Since g is continuous at $f(u_0)$ there is a $\delta_1 > 0$ such that $\gamma(g(v), g(f(u_0))) < \epsilon$ if $\sigma(v, f(u_0)) < \delta_1$. Since f is continuous at u_0 there is a $\delta > 0$ such that $\sigma(f(u), f(u_0)) < \delta_1$ if $\rho(u, u_0) < \delta$. Therefore, $\gamma(h(u), h(u_0)) < \epsilon$ if $\rho(u, u_0) < \delta$.

8.3.6. Let $u_0 \in A$ and suppose that $\epsilon > 0$. Since g is continuous on $\mathbb{R}^n$, there is a $\delta > 0$ such that $|g(y_1, y_2, \dots, y_k) - g(u_0(s_1), u_0(s_2), \dots, u_0(s_k))| < \epsilon$ if $\max\left\{|y_i - u_0(s_i)| \,\middle|\, 1 \le i \le k\right\} < \delta$. Therefore, $|f(u) - f(u_0)| < \epsilon$ if $\rho(u, u_0) < \delta$.

8.3.7. Since $u(s) \ge f(u)$ and $|u(s) - v(s)| \le \rho(u, v)$ for all s in S, $v(s) \ge f(u) - \rho(u, v)$ for all s. Hence, (A) $f(v) \ge f(u) - \rho(u, v)$. If $\epsilon > 0$, there is an s_0 such that $u(s_0) < f(u) + \epsilon/2$. Since $|u(s_0) - v(s_0)| \le \rho(u, v)$, $v(s_0) < f(u) + \epsilon/2 + \rho(u, v)$. Therefore, $f(v) < f(u) + \epsilon/2 + \rho(u, v)$. This and (A) imply that $|f(v) - f(u)| < \rho(u, v) + \epsilon/2$. Therefore, $|f(v) - f(u)| < \epsilon$ if $\rho(u, v) < \epsilon/2$.

Since $u(s) \le g(u)$ and $|u(s) - v(s)| \le \rho(u, v)$ for all s in S, $v(s) \le g(u) + \rho(u, v)$ for all s. Hence, (B) $g(v) \le g(u) + \rho(u, v)$. If $\epsilon > 0$, there is an s_0 such that $u(s_0) > g(u) - \epsilon/2$. Since $|u(s_0) - v(s_0)| \le \rho(u, v)$, $v(s_0) > g(u) - \epsilon/2 + \rho(u, v)$. Therefore, $g(v) > g(u) - \epsilon/2 - \rho(u, v)$. This and (B) imply that $|g(v) - g(u)| < \rho(u, v) - \epsilon/2$. Therefore, $|g(v) - g(u)| < \epsilon$ if $\rho(u, v) < \epsilon/2$.

8.3.8. Since $|f(u) - f(v)| = \left|\int_a^b (u(x) - v(x))\,dx\right| \le \int_a^b |u(x) - v(x)|\,dx \le \rho(u, v)(b - a)$, it follows that $|f(u) - f(v)| < \epsilon$ if $\rho(u, v) < \epsilon/(b - a)$.